

Phase transition in firefly cellular automata on finite trees

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Abstract

We study a one-parameter family of discrete dynamical systems called the κ -color firefly cellular automata, proposed by the author in [13] as discrete models for finite-state pulse-coupled inhibitory oscillators. At each discrete time t , each vertex in a graph has a state (color) in $\{0, \dots, \kappa - 1\}$, and a special color $b(\kappa) = \lfloor \frac{\kappa-1}{2} \rfloor$ is designated as the ‘blinking’ color. During each transition from time t to $t + 1$, each vertex simultaneously increment its color from i to $i + 1$ modulo κ unless $i > b(\kappa)$ and at least one of its neighbors is at the blinking color $b(\kappa)$, in which case it stays at the same color. We say a given initial κ -coloring on vertices of a graph synchronizes if it evolves to a constant coloring so that all vertices increment their color by 1 in unison thereafter. It is easy to see that if every initial κ -coloring on a fixed finite tree synchronizes, then necessarily the tree has maximum degree strictly less than κ . We show that the converse of this statement is true for $\kappa \in \{3, 4, 5, 6\}$, but we also provide counterexamples for $\kappa = 7$ up to 30, exhibiting a phase transition in our model on finite trees “between” $\kappa = 6$ and $\kappa = 7$.

Keywords:

synchronization, coupled oscillators, cellular automaton, finite trees, phase transition

1. Introduction

Many biological complex systems consist of levels of hierarchies of locally interacting dynamic units, whose internal dynamics are induced by non-linear aggregation of local interactions between units at lower levels. Top levels are forced to have a certain macro-behavior suitable for survival, which is miraculously supplied by the right micro-level local interactions, forged by the evolutionary process. This chain of emergent dynamics are at the heart of the challenge we are facing in understanding not only biological systems, but also many other complex systems in our society as well as in designing cooperative control protocol of large networked systems [16], [20].

Consisting of only two levels of hierarchies with simple internal dynamics for units at the bottom level, system of coupled oscillators has been a central subject in non-linear dynamical systems literature for decades [19]. As populations of blinking fireflies [4] and circadian pacemaker cells [7] do, two neighboring oscillators are coupled so that they tend to synchronize their phase or frequency, and the question is that whether such local tendency to synchrony does lead to global synchronization in the entire network. Despite their simplicity they exhibit many fundamental difficulties which repel our traditional reductionist approach based on linear methods, and yet our enhanced knowledge on such systems are finding fruitful applications, ranging from robotic vehicle networks [17] to electric power networks [6], and more recently, to distributed control of wireless sensor networks [11], [18], [22], [21].

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Formulated in a discrete setting, understanding the tight interplay between non-linear local interaction of coupled oscillators and underlying network topology gives a fascinating combinatorial problem. A combinatorial framework on modeling complex systems is called a generalized cellular automaton (GCA), which we describe here. Given a simple connected graph $G = (V, E)$ and a fixed integer $\kappa \geq 2$, the microstate of the system at a given discrete time $t \geq 0$ is given by a κ -coloring of vertices $X_t : V \rightarrow \mathbb{Z}_\kappa = \mathbb{Z}/\kappa\mathbb{Z}$. A given initial coloring X_0 evolves in discrete time via iterating a fixed deterministic *transition map* (or *coupling*) $\tau : X_t \mapsto X_{t+1}$, which depends only on local information at each time step. This generates a trajectory $(X_t)_{t \geq 0}$, and its limiting behavior in relation to the topology of G and the parameter κ is of our interest. Especially, we are interested in finding conditions on G and κ , under which arbitrary initial κ -coloring on G *synchronizes*, i.e., X_t is a constant coloring for all large t .

The problem of designing a GCA model for coupled oscillators which has the capacity to synchronize arbitrary κ -coloring on a class of finite graphs has been known as the digital clock synchronization problem in distributed algorithms literature. If one allows κ to grow with the size of G , then there is such a solution which works on arbitrary finite graphs (e.g., see Dolev [5] or Arora et al. [1]). Roughly speaking, the idea is that if κ is large enough, then one can let every vertex to adapt the locally maximum color within distance 1 at each time step in parallel; then the globally maximum color would propagate and “eat up” all vertices. In fact, the idea of “tuning toward maximum” dates back to a famous consensus algorithm by Lamport [12]. One can readily see that such algorithm relies on some notion of global total ordering among colors of vertices, which is not supplied for fixed κ due to a cyclic nature of the color space. In fact, this issue arising from the cyclic hierarchy between colors is fundamental to our problem, and in fact is a key source which generates interesting emergent behavior in the system. Hence we may restrict ourselves on GCA models with κ independent of G .

Dolev [5] showed that no such GCA model using a fixed κ but synchronizes arbitrary κ -coloring on all connected finite graphs does not exist. Roughly speaking, for any such given GCA model, one can construct a symmetric configuration on a cycle of some length so that the vertices have no way to break such symmetry by blindly following a homogeneous local rule. On trees, however, such a construction is topologically prohibited so one may hope that there exists a κ -color GCA model which synchronizes all initial κ -colorings on any finite trees. Indeed, a 3-color GCA model was studied by Herman and Ghosh [10], and odd $\kappa \geq 3$ models by Boulunier, Petit, and Villain [2]. When $\kappa = 3$, the latter model coincides with another well-known GCA model called the cyclic cellular automaton, which was introduced by Bramson and Griffeath [3] as a discrete time analogue of the cyclic particle systems. In recent work with Gravner and Sivakoff [8], we studied the limiting behavior of 3-color cyclic cellular automaton together with the 3-color Greenberg-Hastings model [9] on infinite trees using probabilistic methods.

The model we are interested in the present work is a one-parameter family of GCAs which we call the firefly cellular automata (FCAs), proposed by the author in a recent work [13] as a discrete model for pulse-coupled inhibitory oscillators. The model is defined for each integer $\kappa \geq 3$. Among κ possible colors for each vertices, a special state $b(\kappa) = \lfloor \frac{\kappa-1}{2} \rfloor$ is designated as the ‘blinking’ color. In a network of κ -state identical oscillators, each oscillator updates from state i to $i + 1 \pmod{\kappa}$ unless it sees a neighbor and notices that its phase is ahead of the blinking neighbor, in which case it waits for one iteration without update. More precisely, the transition map $\tau : X_t \mapsto X_{t+1}$ for the κ -color FCA is given as follows:

$$(FCA) \quad X_{t+1}(v) = \begin{cases} X_t(v) & \text{if } X_t(v) > b(\kappa) \text{ and } |\{u \in N(v) : X_t(u) = b(\kappa)\}| \geq 1 \\ X_t(v) + 1 \pmod{\kappa} & \text{otherwise} \end{cases} \quad (1)$$

We call a unit of time a “second”. We say a vertex v *blinks* at time t if $X_t(v) = b(\kappa)$, is *pulled*

at time t if $X_{t+1}(v) = X_t(v)$, and *pulls* its neighbor u at time t if u is pulled at time t and v blinks at time t . Given a κ -color FCA trajectory $(X_t)_{t \geq 0}$ on a graph $G = (V, E)$, we say X_t (or X_0) *synchronizes* if there exists $N \geq 0$ such that $X_t \equiv \text{Const.}$ for all $t > N$.

Being a deterministic dynamical system with finite state space for each vertex, any κ -color FCA trajectory $(X_t)_{t \geq 0}$ on any finite graph $G = (V, E)$ must converge to a periodic limit cycle. Limit cycles can be either a synchronous or asynchronous periodic orbit, as illustrated in the examples of 6-color FCA trajectories in Figures 1. Note that $b(6) = 2$ is the blinking state in this case, so every vertex of state 3, 4, or 5 with a state 2 neighbor stops evolving for 1 second and all the other vertices evolves to the next state.

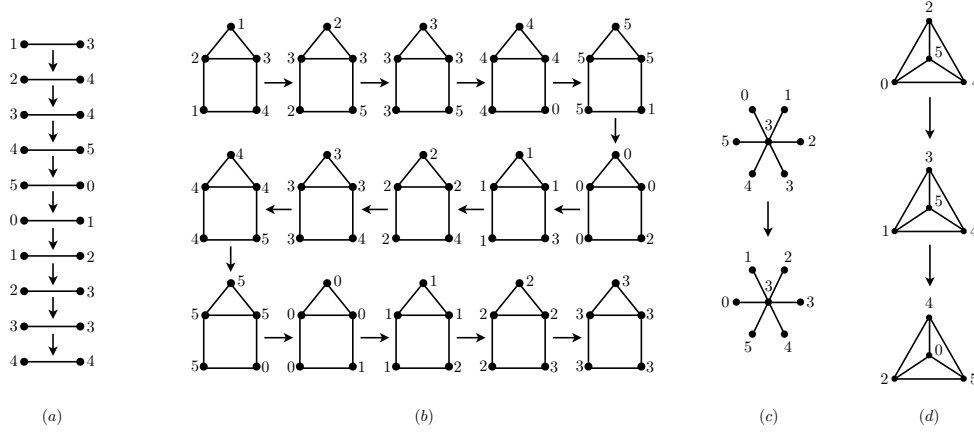


Figure 1: Two examples of synchronizing 6-color FCA trajectories are shown in (a) and (b). In (c) and (d), the last configurations are symmetric to the initial ones, so the networks do not synchronize.

In [13], we have shown that for any $\kappa \geq 3$, arbitrary κ -coloring on finite paths synchronizes in finite time (Theorem 2). This result is pushed further on infinite paths in a joint work with David Sivakoff [15], in the sense that if the initial κ -coloring on the integer lattice \mathbb{Z} is given at random according to the uniform product measure, then the probability that there is only one color on a fixed finite interval at time t converges to 1. On finite trees, however, one does not have such a universal synchronization behavior for all $\kappa \geq 3$ in general. As illustrated by example (c) in Figure 1, there exists a tree with a non-synchronizing 6-configuration. The obstruction there is that the center of a star with many leaves could be delayed by the leaves constantly. In general, let v be a vertex in finite tree T with degree $\geq \kappa$, and let T_1, \dots, T_m be the connected components of $T - v$, the graph obtained from T by deleting v together with edges incident to it. Note that $m \geq \kappa$. Assign state $i \pmod{\kappa}$ to every vertex of T_i , and assign any state $> \kappa/2$ to vertex v . Then v never blinks and each component T_i never get pulled by v , which is essentially the counterexample in Figure 1 (c). Therefore if every n -configuration on T synchronizes, then necessarily T has maximum degree $< \kappa$.

Our goal in this paper is to establish that such a necessary local condition to synchronize arbitrary κ -coloring on a tree is also sufficient for $\kappa \in \{3, 4, 5, 6\}$, but not necessarily for $\kappa \geq 7$:

Theorem 3. *Let $T = (V, E)$ be a finite tree and let $\kappa \geq 3$.*

- (i) *Suppose $\kappa \in \{3, 4, 5, 6\}$. Then arbitrary κ -coloring $X_0 : V \rightarrow \mathbb{Z}_\kappa$ synchronizes if and only if T has maximum degree $< \kappa$.*
- (ii) *For each $\kappa \geq 7$, there exists a finite tree T with maximum degree $< \kappa$ and a non-synchronizing κ -coloring X_0 .*

In other words, the above theorem claims the κ -color FCA on finite trees has a phase transition “between” $\kappa = 6$ and 7. For part (ii) of Theorem 3, we provide examples of non-synchronizing network on trees with small maximum degree for $\kappa = 7$ up to 30, which are given in the appendix. Such examples are constructed on stars with $\leq \kappa - 1$ leaves for all but $\kappa = 8$; for $\kappa = 8$ every 8-coloring on stars with < 8 leaves synchronize, but we give a non-synchronizing example on a tree obtained by joining two 3-stars by adding an edge between their centers. Our list of non-synchronizing examples does not seem to have an apparent pattern which would allow us to systematically construct such examples for all $\kappa \geq 7$, but computer simulation seems to indicate that this phenomenon continues to hold for all $\kappa \geq 7$.

On the other hand, part (i) of the above theorem for $\kappa \in \{3, 4, 5\}$ was shown in [13]. Roughly speaking, the idea is to classify possible local dynamics on a small substructure of trees and then show that the dynamics restricts onto a proper subtree. Namely, take an induced subgraph $H \subset T$ such that there is only one edge between H and its complement. Only a single vertex in H could be perturbed (pulled) from outside, and since the coupling between vertices in FCA is inhibitory, such perturbation cannot occur more than once in every κ seconds. Thus If we understand possible dynamics on H under such a sporadic perturbation, then we may be able to reduce the dynamics on T onto a proper subtree and use induction. Investigating such local dynamics would be easier if we take H as small as possible, and for $\kappa \in \{3, 4, 5\}$ this could be done in a relatively clean manner. But “near the criticality” for $\kappa = 6$, the combinatorics gets substantially more involved and classifying local dynamics alone does not allow us to restrict global dynamics on some proper subtree. Proving of part (i) of the above theorem for $\kappa = 6$ is the main theme of this paper. In fact, we prove the $\kappa = 6$ case of the following stronger statement:

Theorem 4. *Let T be a finite tree and let X_0 be a κ -coloring on T for $\kappa \in \{3, 4, 5, 6\}$. Then X_0 synchronizes if and only if every vertex of T blinks infinitely often in the dynamic $(X_t)_{t \geq 0}$.*

It is indeed stronger than the original statement since assuming the maximum degree of T is at most 5 guarantees its hypothesis:

Lemma 5. *Let $G = (V, E)$ be a graph and let u be a vertex. Suppose $\deg_G(u) < \kappa$. Let X_0 be any κ -coloring on G . Then u blinks infinitely often in the dynamic $(X_t)_{t \geq 0}$.*

The proof of Lemma 5 is omitted in this paper but can be found in [13].

We remark that the emphasis on our main result in the present work is not on its algorithmic applications to clock synchronization problems (in that direction, see our recent work on an asynchronous generalization of 4-color FCA in [14]). Rather, it is the combinatorial problem itself inspired by non-linear dynamics of coupled oscillators and related problems, and the curious phase transition behavior in our model.

This paper is organized as follows. We first recall some basic facts and lemmas about FCA we have established in [13] in Section 2. Then we give an outline of the Proof of Theorem 4 for $\kappa = 6$ in Section 3, together with its proof assuming two key lemmas. In general we assume the existence of a minimal counterexample to the theorem and analyze its properties, and obtain contradiction. Section 4 is devoted to a preliminary analysis on local dynamics of such minimal counterexample. In subsequent sections, Section 5 and 6, we prove the two key lemmas and complete the proof of Theorem 4 for $\kappa = 6$. In the appendix, we give a (non-exhaustive) list of counterexamples supporting Theorem 3 (ii) for $\kappa \in [7, 30]$.

2. Generalities and the branch width lemma

Throughout in this paper we assume every graph is finite, simple, and connected, unless otherwise mentioned. If S, H are vertex-disjoint subgraphs of G , then $S + H$ is defined by the subgraph obtained from $S \cup H$ by adding all the edges in G between S and H . On the other hand, $S - H := S - V(H)$ denotes the subgraph of G obtained from S by deleting all vertices of H and edges incident to them. If $v \in V(H)$ and H is a subgraph of G , then $N_H(v)$ denotes the set of all neighbors of v in H and $\deg_H(v) := |N_H(v)|$.

It will be convenient to introduce a geometric representation of the FCA dynamics. Let $(X_t)_{t \geq 0}$ be a κ -color FCA dynamics on a graph $G = (V, E)$. The idea is to consider the induced dynamics $(Y_t)_{t \geq 0}$, where Y_t is the *relative configuration* given by

$$Y_t = X_t - t + 2 \pmod{\kappa}. \quad (2)$$

We refer to the value $Y_t(v)$ the *phase* of v at time t . Note that in the original dynamics, a node blinks whenever $X_t = b(\kappa)$, so in the relative dynamics $(Y_t)_{t \geq 0}$, a node blinks whenever it has phase $-t \pmod{\kappa}$. In this relative dynamics vertices keep the same phase until they get pulled, in which case they decrease their phase by 1. A comparison between the original dynamics and the relative dynamics in case $\kappa = 6$ is illustrated by example in Figure 2. The geometric representation of the relative FCA dynamics in the second row in Figure 2 is what we call the *relative circular representation*. The hexagon represents the phase space, which is the original color space \mathbb{Z}_6 modulo rotation, increasing in clockwise orientation. The open circle inside it, called the *activator*, revolves around the phase space counterclockwise at unit speed, whose location at time t is $-t \pmod{6}$. Hence whenever a vertex has the same phase as the activator, the node blinks.

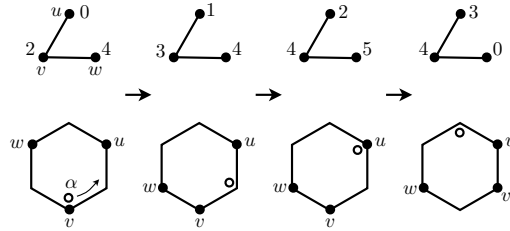


Figure 2: A comparison between the standard and relative circular representation of a 6-color FCA dynamics. The latter lacks adjacency information between nodes. For example, in the third configuration, u only pulls v since it is not adjacent to w . If u were adjacent to w , then it would have pulled w as well.

Let u, v be two vertices in G . The *clockwise displacement of v from u at time t* is defined by

$$\delta_t(u, v) := X_t(v) - X_t(u) \pmod{\kappa}. \quad (3)$$

We say v is *clockwise* to u and u is *counterclockwise* to v at t if $\delta_t(u, v) < n/2$, and u is *opposite* to v if $\delta_t(u, v) = n/2$, which can happen only if n is even. Suppose u and v are adjacent in G . We say v is a *clockwise neighbor* of u at t if v is clockwise to u at t , and *counterclockwise neighbor* at t otherwise. The *width* of a X_t (or Y_t) is defined to be the quantity

$$w(X_t) := \min_{v \in V} \max_{u \in V} \delta_t(u, v),$$

which is the length of the shortest path on the color space \mathbb{Z}_n (viewed as a cycle of length n) that covers all states of the vertices in the configuration. For instance, the first configuration

in Figure 2 has width 4, whereas the last one has width 3. For any subgraph $B \subset G$, we denote by $w_B(X)$ the width of the restricted configuration $X|_{V(B)}$ on B .

A classic observation in the theory of pulse-coupled oscillators is that the width $w(X_t)$ at time t converges to 0 monotonically if $w(X_0) < \kappa/2$. Roughly speaking, the intuition is that if at some point the width at time $t = s$ is strictly less than half of the perimeter of color space \mathbb{Z}_κ , then one can define a global total ordering on all occupied phases at time $t = s$ from the most lagging to the most advancing. Under a very mild condition on the coupling, this total ordering is respected by the dynamics and the farthest displacement monotonically decreases. For more details see Lemma 2.2 and following discussions in [13]. Its key mechanism is illustrated in the example in Figure 3.

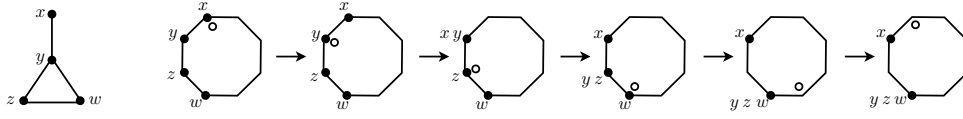


Figure 3: An example illustrating the proof of width lemma when $n = 8$ and $G = B$. Because of the small width condition on B , the head vertices including x does not affect the tail vertices including w , and all vertices tend to move toward w . The finite size and connection of G will then give the assertion. It is important to note that x never pulls y .

A natural extension of the above observation to a proper subgraph is our starting point to understand FCA dynamics on finite trees, which we shall introduce now. A connected subgraph $S \subseteq G$ is called a k -star if it has a vertex v , called the *center*, such that all the other vertices of S are leaves in G . A k -star S is called a k -branch if the center of S has only one neighbor in $G - S$, which we may call the *root* of S . We may denote a k -branch by B rather than by S . Note that branches are smallest induced subgraphs of trees with a single vertex adjacent to its complement. Hence it is the smallest subgraph which gets minimal perturbation from outside and yet it should have a simple internal dynamics. Our observation is that if the width restricted on a branch B , which we call the *branch width*, is strictly smaller than $\kappa/2 - 1$, then such small branch width is maintained in the dynamics and the global dynamics restricts on the complement $G - B$.

We say a dynamic $(X_t)_{t \geq 0}$ on G *restricts on* $H \subset G$ if the restriction $X_t \mapsto X_t|_H$ and transition map τ commute, i.e., the induced restricted dynamic $(X_t|_H)_{t \geq 0}$ follows the same transition map on H . We say the dynamic $(X_t)_{t \geq 1}$ on G *restricts on* H *eventually* if there exists $r \geq 0$ such that $(X_t)_{t \geq r}$ restricts on H .

Lemma 2.1 (branch width lemma). *Let $G = (V, E)$ be a graph with a k -branch B rooted at some vertex $w \in V$. Let u be the center of B and $l_1, \dots, l_k, k \geq 1$ be its leaves. Let H be the graph obtained from G by deleting the leaves of this branch. Let X_0 be a κ -coloring on G for any $\kappa \geq 3$ with $w_B(X_0) < \kappa/2 - 1$. Then we have the followings:*

- (i) u is clockwise to all leaves of B at some time $r \leq n(w_B(X_0) + 1)$ and $w_B(X_r) \leq w_B(X_0)$;
- (ii) u is clockwise to all leaves of B for all $t \geq r$, and $w_B(X_t) \leq w_B(X_0) + 1$ for all $t \geq 0$;
- (iii) If u is clockwise to all leaves at $t = r$, then the dynamic $(X_t)_{t \geq r}$ restricts on H ;
- (iv) If every κ -coloring on H synchronizes, then X_0 synchronizes.

A detailed proof can be found in [13], and here we give a brief sketch through an example. Suppose $\kappa = 8$ and $k = 3$. Since the coupling is inhibitory, the leaves of B and the root w only pulls u until it becomes the most lagging one in B . So eventually, we will have a situation as in the first diagram in Figure 5, where the branch width w_B is still strictly less than $\kappa/2 - 1$ and the center u is at most lagging in B . Now the root w pulls v at most once in every κ seconds,

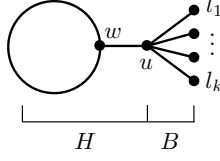


Figure 4: A graph G with a k -branch B

increasing the branch width by 1. But since we have a wiggle room on the branch width, the increased branch width is still small ($< \kappa/2$) and the leaves do not pull u until its next blink. Then the center u blinks and pulls all leaves, decreasing the branch width by 1. Hence the original branch width is recovered, and this scenario repeats over and over again. In this cycle the leaves never pull the center, so the dynamics restricts on H .

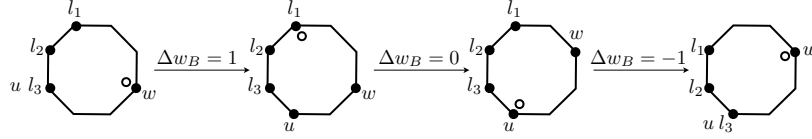


Figure 5: An illustration of branch width recovery for $n = 8$ and $k = 3$. Once u is the most lagging, w can pull u to increase the branch width by 1 but u pulls the most advancing leaves and decrease the branch width by 1, before w blinks again. Note that w could get external pulls from its neighbors different from u but it doesn't affect our argument.

3. Outline of the proof of Theorem 4 for $\kappa = 6$.

Trees have nice recursive property when viewed as *rooted trees*. A rooted tree $T = (V, E)$ is a tree with a designated vertex r called the *root*. For each $v \in V$, let \vec{P}_v be the unique path from r to v , and let v^- be the unique neighbor of v in \vec{P}_v . For distinct vertices u, v in T , we say u is the *parent* of v and v is a *child* of u if $u = v^-$. Write $u \leq v$ if $u \in V(\vec{P}_v)$. We say u is an *ancestor* of v and v is a *descendant* of u if $u \leq v$. For each $v \in V$, we define the *descendent subtree* T_v to be the subtree of T consisting of all descendants of v . The *depth* of T_v , denoted by $\text{dep}(v)$, is the maximum level of vertices in T_v . We say a descendent subtree T_v is a *terminal branch* if it is a branch and either $T_v = T$, or $v^- \in V$ and for all children u of v^- , T_u is either a singleton or a branch. Note that any rooted tree with depth ≥ 2 has at least one terminal branch.

Now for each $\kappa \geq 3$, call a pair (T, X_0) of a finite tree and a κ -coloring on it a *minimal counterexample* if (1) X_0 does not synchronize, (2) every vertex in T blinks infinitely often in the dynamic $(X_t)_{t \geq 0}$, and (3) $|V(T)| \leq |V(T')|$ if (T', X'_0) is another pair with satisfying (1) and (2). We may assume without loss of generality that $(X_t)_{t \geq 0}$ is periodic, by choosing X_0 from the periodic limit cycle. Note that by the minimality of T and Lemma 2.1, every branch in a minimal counterexample must have branch width $\geq \kappa/2 - 1$ for all times. This enforces a very specific local dynamics on branches which easily led to contradiction in case of $\kappa \in \{3, 4, 5\}$. Namely, let (T, X_0) be a minimal counterexample for $\kappa \in \{3, 4, 5\}$. In [13], we have proceeded as follows:

$\kappa = 3$. Every branch in T eventually has small branch width, a contradiction;

$\kappa = 5$. If B is a branch in a T , then the dynamics restricts eventually restricts on T less the leaves of B , a contradiction;

$\kappa = 4$. If T_v is a terminal branch in T , then the dynamics restricts on $T - T_v$ eventually, a contradiction.

The argument for $\kappa = 4$ is notable. Suppose w is a vertex in T such that each descendant tree T_v at its children is either a leaf or a branch (e.g., take w to be the parent of the center of a terminal branch). Each subtree T_v must have one of a few local dynamics enforced from the minimality, which gives some constraint on the local dynamic on w . Roughly speaking, the contradiction is obtained by showing that such constraints from multiple components are not compatible. Unfortunately, however, for $\kappa = 6$, the enforced dynamics on branches still has large entropy and one should consider the ensemble of all possible local dynamics joining at the common root. Moreover, analyzing enforced dynamics on depth 2 descendant trees is not enough; in fact, we have to go all the way down to the root to get a contradiction.

Let (T, X_0) be a minimal counterexample for $\kappa = 6$, and we fix this notation hereafter throughout later sections. Let T_v be a proper descendant subtree in T . We say T_v is *open* if the induced dynamics on T_v requires v to be pulled by its parent v^- . In particular, the whole tree $T = T_x$ cannot be open. If T_v is open, then the minimality will force particular induced local dynamics on its root v^- . To represent induced local dynamics on a single vertex concisely, we introduce the following notion. For each vertex v in T , let $\text{bt}_i(v)$ be the time of i th blink of v in the orbit, for each $i \geq 1$. The i th blinking gap is given by $g_i(v) = \text{bt}_{i+1}(v) - \text{bt}_i(v)$. The *blinking sequence* of v is defined by the sequence $(g_i(v))_{i \geq 1}$ of blinking gaps of v . Note that since we are in a periodic orbit, the blinking sequence of any $v \in V$ repeats a finite sequence of blinking gaps. There will be mainly four different types of enforced dynamics on T_v as we define below:

Definition 3.1. Let (T, X_0) as before and let T_v an open descendant subtree of T .

- (i) We say T_v is of type (a) iff $g_i(v) \equiv 12$ and $2 \in \{X_{t+4}(v^-), X_{t+5}(v^-)\}$ whenever $X_t(v) = 2$;
- (ii) We say T_v is of type (b) iff $g_i(v)$ alternates 9 and 7, and $2 \in \{X_{t+2}(v^-), X_{t+4}(v^-)\}$ whenever $X_t(v) = X_{t+9}(v) = 2$.
- (iii) We say T_v is fractal of type 10/9 iff $g_i(v)$ alternates 10 and 9, and $2 \in \{X_{t+1}(v^-), X_{t+4}(v^-)\}$ whenever $X_t(v) = X_{t+10}(v) = 2$;
- (iv) We say T_v is fractal of type 11/8 iff $g_i(v)$ alternates 11 and 8, and $2 \in \{X_{t+2}(v^-), X_{t+4}(v^-)\}$ whenever $X_t(v) = X_{t+11}(v) = 2$.

We say T_v is *fractal* if its fractal of either types.

Below we give a more direct characterization of type (a), (b), and fractal subtrees in terms of the induced dynamics on v and its parent v^- . Suppose a descendant subtree T_v is of type (a). From the definition, it is easy to see that the local dynamics on v and v^- are given by concatenating the following four sequences:

$$\begin{array}{cccccccccccccccc} v & 2 & 3 & - & - & - & - & - & - & - & 5 & 0 & 1 & 2 \\ v^- & 5 & 5 & 0 & 1 & 2 & 3 & - & - & - & - & - & - & - \end{array} \quad (\text{P})$$

$$\begin{array}{cccccccccccccccc} v & 2 & 3 & - & - & - & - & - & - & - & 5 & 0 & 1 & 2 \\ v^- & 4 & 4 & 5 & 0 & 1 & 2 & 3 & - & - & - & - & - & - \end{array} \quad (\text{Q})$$

$$\begin{array}{cccccccccccccccc} v & 2 & 3 & - & - & - & - & - & - & - & 5 & 0 & 1 & 2 \\ v^- & 5 & 5 & 5 & 0 & 1 & 2 & 3 & - & - & - & - & - & - \end{array} \quad (\text{R})$$

$$\begin{array}{cccccccccccccccc} v & 2 & 3 & - & - & - & - & - & - & - & 5 & 0 & 1 & 2 \\ v^- & 5 & 5 & 0 & 1 & 2 & 3 & 4 & 5 & 0 & 1 & 2 & 3 & 4 \end{array} \quad (\text{S})$$

where time goes from left to right and none of $-$'s are 2. Each of the above sequences describe dynamics on ν and ν^- for 12 iterations, and the sequence (P)(P) obtained by concatenating the sequence (P) twice describes 24 iterations, for instance. The four possibilities came from considering possible instances of ν^- blinking after its first blink during each blinking gap 12 of ν .

Similarly, if T_ν is of *type (b)*, then the local dynamics on ν and ν^- are given by concatenating the following six sequences:

$$\begin{array}{cccccccccccccccc} \nu & 2 & 3 & - & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & 5 & 0 & 1 & 2 \\ \nu^- & 0 & 1 & 2 & 3 & - & - & - & - & - & - & - & - & - & - & - & - & - \end{array} \quad (I)$$

$$\begin{array}{cccccccccccccccc} \nu & 2 & 3 & - & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & 5 & 0 & 1 & 2 \\ \nu^- & 5 & 5 & 0 & 1 & 2 & 3 & - & - & - & - & - & - & - & - & - & - & - \end{array} \quad (J)$$

$$\begin{array}{cccccccccccccccc} \nu & 2 & 3 & - & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & 5 & 0 & 1 & 2 \\ \nu^- & 0 & 1 & 2 & 3 & 4 & 5 & 0 & 1 & 2 & 3 & - & - & - & - & - & - & - \end{array} \quad (X)$$

$$\begin{array}{cccccccccccccccc} \nu & 2 & 3 & - & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & 5 & 0 & 1 & 2 \\ \nu^- & 0 & 1 & 2 & 3 & - & - & 5 & 0 & 1 & 2 & 3 & - & - & - & - & - & - \end{array} \quad (Y)$$

$$\begin{array}{cccccccccccccccc} \nu & 2 & 3 & - & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & 5 & 0 & 1 & 2 \\ \nu^- & 0 & 1 & 2 & 3 & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & - & - & - \end{array} \quad (Z)$$

$$\begin{array}{cccccccccccccccc} \nu & 2 & 3 & - & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & 5 & 0 & 1 & 2 \\ \nu^- & 5 & 5 & 0 & 1 & 2 & 3 & - & - & 5 & 0 & 1 & 2 & 3 & - & - & - & - \end{array} \quad (W)$$

where none of $-$'s are 2, as before.

Finally, the same holds for the following two sequences

$$\begin{array}{cccccccccccccccccccc} \nu & 2 & 3 & 3 & - & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & - & - & 5 & 0 & 1 & 2 \\ \nu^- & 1 & 2 & 3 & - & - & - & - & a_1 & a_2 & a_3 & a_4 & a_5 & - & a_7 & a_8 & - & - & - & - \end{array} \quad (F1)$$

$$\begin{array}{cccccccccccccccccccc} \nu & 2 & 3 & - & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & - & - & 5 & 0 & 1 & 2 \\ \nu^- & 5 & 5 & 0 & 1 & 2 & 3 & - & - & - & - & b_3 & b_4 & b_5 & - & b_7 & b_8 & - & - & - \end{array} \quad (F2)$$

when T_ν is fractal of type 10/9, and with the following two sequences for T_ν fractal of type 11/8:

$$\begin{array}{cccccccccccccccccccc} \nu & 2 & 3 & - & - & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & - & 5 & 0 & 1 & 2 \\ \nu^- & 0 & 1 & 2 & 3 & - & - & - & - & c_2 & c_3 & c_4 & c_5 & c_6 & - & c_8 & - & - & - & - \end{array} \quad (F3)$$

$$\begin{array}{cccccccccccccccccccc} \nu & 2 & 3 & - & - & - & - & 5 & 0 & 1 & 2 & 3 & - & - & - & 5 & 0 & 1 & 2 \\ \nu^- & 5 & 5 & 0 & 1 & 2 & 3 & - & - & - & - & d_3 & d_4 & d_5 & d_6 & - & d_8 & - & - & - \end{array} \quad (F4)$$

As before, none of $-$'s are 2 in any of the sequences above, but other instances a_i 's, b_i 's, c_i 's, and d_i 's could be 2. We could specify all sequences when ν^- could blink among those instances, but there would be too many cases in doing so.

Now we outline the proof of Theorem 4 for $\kappa = 6$. In a nutshell, we show that every proper descendant subtree T_ν of depth ≥ 1 is fractal. In particular, every component in $T_\nu - r$ will be either a singleton or fractal. A recursive property of fractal subtrees would then yield that the whole tree is fractal, and in particular, open. This contradiction shows that minimal counterexample for $\kappa = 6$ does not exist. To give more detail, we first show by using Lemma 2.1, that every branch must be open and of type (a) or (b), or fractal of type 10/9. Furthermore, we will show that if T_ν is a terminal branch, then it cannot be of type (a) or (b), as stated in the following lemma:

Lemma 2.3. *Let (T, X_0) be as before. Then every terminal branch of T is fractal.*

Next, the induction step is based on the recursive property of fractal branches stated in the following lemma:

Lemma 2.4. *Let (T, X_0) be as before. Let $w \in V$ and suppose that each connected component of $T_w - w$ is either a singleton or a fractal branch. Then T_w is open and fractal. In particular, $w^- \in V(T)$.*

These two lemmas are enough to prove our main theorem.

Proof of Theorem 4 for $\kappa = 6$. It suffices to show the "if" part. For the contrary, suppose there exists a minimal counterexample (T, X_0) for $\kappa = 6$. Choose a vertex r and view T as being rooted at r . By minimality, the depth of $T = T_r$ is at least 1. If $\text{dep}(r) = 1$, then $T = T_r$ is a terminal branch, which is open by Proposition 4.4, a contradiction. Hence we may assume that $\text{dep}(r) \geq 2$.

It suffices to show that for every non-leaf and non-root vertex v , T_v is fractal. Indeed, this would yield that $T_r - r$ is a disjoint union of leaves and fractal subtrees, but by Lemma 2.4 T_r must be fractal, a contradiction. We proceed by an induction on $\text{dep}(v^-) \geq 2$. For the base step, note that $\text{dep}(v^-) = 2$ means that $T_{v^-} - v^-$ is a disjoint union of leaves and terminal branches. Since by Lemma 2.3 terminal branches are fractal, Lemma 2.4 gives that T_v is fractal. The induction step follows similarly. If $\text{dep}(v^-) = d \geq 3$, then $T_v - v$ is a disjoint union of leaves and depth $< d - 1$ descendant subtrees. By induction hypothesis and Lemma 2.4, T_v must be fractal. This shows the assertion. ■

4. Analysis of enforced local orbits on branches

Throughout this section, (T, X_0) is a minimal counterexample for $\kappa = 6$. For each descendant subtree T_v , the minimality forces a particular local dynamic on T_v , which may give some constraints on its root v^- . In this section, we analyze such enforced local orbits on T_v and see how they restrict the dynamics on v^- when T_v is either a leaf, branch, or a fractal branch. Furthermore, we investigate possible ensemble of such constraint on the local dynamics of v^- when it has multiple descendant subtrees rooted at itself. A conceptual background is a classic technique in dynamical systems literature called the Poincaré return map, which is to look at transitions between snapshots of system configuration where a particular vertex takes a particular state. We adapt this concept in a local setting: we consider all possible local configurations on a descendant subtree T_v in which v blinks. Since we are assuming that v blinks infinitely often in the dynamic, the global periodic orbit $(X_t)_{t \geq 0}$ must induce a periodic orbit on such special local configurations, together with constraints on the local dynamics on v^- .

We will rely heavily on diagrammatic analysis to study possible blinking sequences of v^- and their ensemble. We shall represent local dynamics on T_v often as a weighted digraph, in which edge weights represent blinking gaps of v and nodes could be snapshots of local configurations or a finite sequence of local dynamics. Let us first introduce some terminologies. Let $D = (V, \bar{E})$ be a digraph with vertex and edge weights $\omega : V \sqcup E \rightarrow \mathbb{N} \cup \{0\}$. We say a sequence (a_n) of positive integers is *generated* by D if there exists a directed walk $P = v_1 e_1 v_2 e_2 \dots$ in D such that (a_n) can be obtained from the sequence $\omega(v_1), \omega(e_1), \omega(v_2), \dots$ by dropping the zero terms. For example, consider a digraph D with vertex set $V = \{X, Y\}$ and edge set $E = \{(XX), (XY), (YX), (YY)\}$ with weights given as in Figure 19.

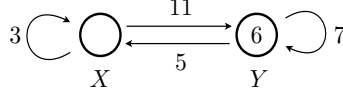


Figure 6: A digraph that can generate finite sequence 3, 11, 6, 7, 6, 5, 3.

Whenever a vertex has weight 0, we shall omit the weight in the diagram. Notice that the directed walk $P = X, (XX), X, (XY), Y, (YY), Y, (YX), X, (XX)$ gives the sequence of weights 0, 3, 11, 6, 7, 6, 5, 0, 3. By dropping out the zero terms, we see that the given digraph can generate the sequence 3, 11, 6, 7, 6, 5, 3.

A first example comes from analyzing local dynamics on a vertex with a leaf neighbor. We begin with a simple example.

Proposition 4.1. *Let (T, X_0) as before. Let v be a vertex in T with a leaf neighbor u . Then u is never opposite to v whenever v blinks. That is, $X_t(u) \neq 5$ if $X_t(v) = 2$.*

Proof. Back-tracking three iterations from such local configuration leads to contradiction, as below:

$$v | u \quad 2 | 5 \quad \leftarrow \quad 1 | 4 \quad \leftarrow \quad 0 | 3 \quad \overset{x}{\leftarrow} \quad 5 | 2, \quad (4)$$

as blinking u must have been pulled v at color 5. \square

Proposition 4.2. *Let (T, X_0) as before. Let v be a vertex in T with a leaf neighbor u . Then the blinking sequence of v is given by $(a_i + 6k_i)_{i \geq 1}$ where $(a_i)_{i \geq 1}$ is generated by the digraph in Figure 7 and $(k_i)_i$ is some sequence of non-negative integers which depend on the dynamics.*

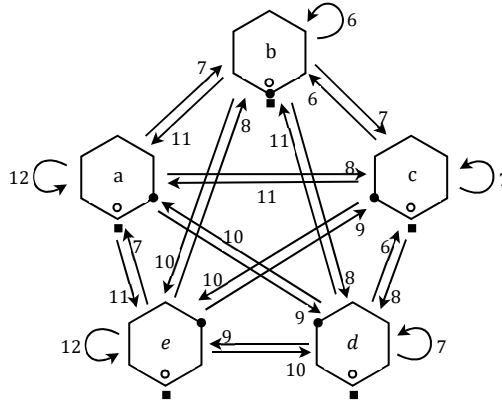


Figure 7: A weighted digraph on five possible non-opposite local configurations on the 1-star $v + u$. ■ = phase of center v , • = phase of leaf u , and ○ = phase of activator. The blinking gap of v corresponding each transition is given by $a + 6k$ where a the edge weight and k is some nonnegative integer depending on the structure and dynamics on G .

Proof. There are five local configurations on the 1-star $v + u$ with center v where v blinks such that u is not opposite to v . The above digraph shows every possible transition between such five non-opposite local configurations. For the edge weights, note that since v is the only neighbor of the leaf u , once v blinks, u maintains its phase until the next blink of v . This determines the blinking gap of v during each transition in the above digraph modulo 6.

For example, consider the transition $d \rightarrow e$ in Figure 7, which is shown in Figure 8.

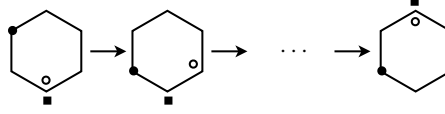


Figure 8: The transition $d \rightarrow e$ in Figure 7. The blinking gap must be $9 + 6k$ for some nonnegative integer k .

Since the center pulls the leaf initially, the phase of leaf moves one step clockwise after the first iteration. Now the leaf does not move until the next blink of the center, so to get the bottom left local configuration in Figure 16, the center must be at the top of the hexagon by the time it blinks again for the first time. Hence looking that the initial and terminal phase of the activator, we conclude that the blinking gap of v during this transition is 9 modulo 6. Other edge weights are determined in similar way. This shows the assertion. \square

Next, we analyze forced local dynamics on branches. In [13] Lemma 3.2, we showed that 1-branches eventually gets small branch width and contradicts the minimality by Lemma 2.1. Hence T does not have a 1-branch. Moreover, if B is any k -branch in T , then all the k leaves there should maintain distinct colors for all times, since otherwise we can delete some leaves and restrict the dynamics on T on a proper subtree, which contradicts the minimality. The following proposition gives how the blinking sequence of a vertex v is restricted if it has multiple leaves, which includes the case when v is a center of a branch in T . Its proof is given at the end of this section.

Proposition 4.3. *Let (T, X_0) be as before. Suppose T has a k -star S for $k \geq 2$ with center v . Then we have the followings:*

- (i) *The induced local dynamics on S is given by one of the four digraphs in Figure 9. In particular, the blinking sequence of v is given by $(a_i + 6k_i)_{i \geq 1}$ where $(a_i)_{i \geq 1}$ is generated by the digraph in Figure 9 and $(k_i)_i$ is some sequence of non-negative integers which depend on the dynamics.*
- (ii) *If $S = T_v$ is a branch, then the induced local dynamics on T_v only uses the five shaded local configuration in Figure 9.*
- (iii) *If $S = T_v$ has local dynamics given by Figure 9 (a), (b), or (c), then it is open and of type (a), (b), or fractal of type 10/9, respectively.*

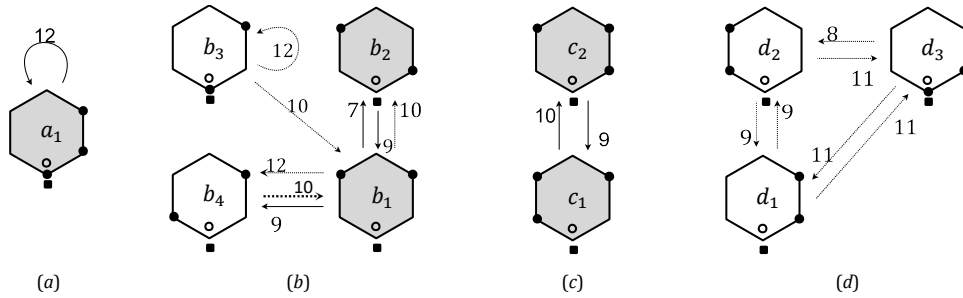


Figure 9: Possible transitions between local configurations on a star with ≥ 2 leaves. The edge weights indicate the corresponding minimal blinking gaps of the center. \blacksquare = phase of center v , \bullet = phase of leaves, and \circ = phase of activator. The dotted transitions are impossible if $B = T_v$ is the center of a branch.

Next, we investigate how the three types of closed orbits on a branch restricts the blinking sequence of its root. Let $(a_n), (b_n)$ be two sequences of real numbers. We say the sequence

(b_n) refines (a_n) and (b_n) is a coarsening of (a_n) if there exists an increasing sequence (d_n) of natural numbers such that

$$a_n = \sum_{d_n \leq k < d_{n+1}} b_k.$$

For instance, the sequence $1, 2, 3, 4, \dots$ refines $3, 7, 11, 15, \dots$ since $3 = 1 + 2$, $7 = 3 + 4$, $11 = 5 + 6$, and so on.

Proposition 4.4. *Let (T, X_0) be as before. Let T_v be a branch in T with $v^- \in T$. Then we have the followings:*

- (i) If T_v is of type (a), then the blinking sequence of v^- is generated by the digraph (A) in Figure 10.
- (ii) If T_v is of type (b) then the blinking sequence of v^- refines a sequence generated by digraph(B1) in Figure 10.
- (iii) In case of (ii), the blinking sequence (g_i) of w is refined by some sequence (b_m) generated by diagram (B2) in Figure 10. Furthermore, $(g_i)_{i \geq 1}$ can be obtained from (b_m) by merging some consecutive terms b_m, b_{m+1} into $b_m + b_{m+1}$, where b_m is a vertex weight and b_{m+1} is following edge weight.
- (iv) In case of (iii), the sequence (b_m) cannot be generated by a closed walk on (B2) which only uses nodes Y or Z .

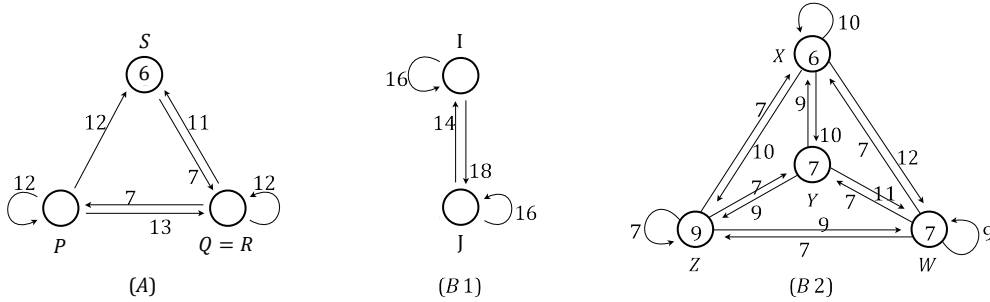


Figure 10: (A) a digraph generating the blinking sequence of the root v^- of type (a) branches; (B1) a digraph generating a coarsening of blinking sequence of v^- of type (b) branches; (B2) a digraph generating a refinement of blinking sequence of v^- of type (b) branches.

Proof. The proof follows mostly from definitions. Let T_v is of type (a). Then concatenating sequences $(P)-(S)$ gives a complete description of the blinking sequence of v^- . For instance, if string $(P)(P)(Q)$ is used in the local dynamics, then v^- blinks exactly once in sequence (P) , and blinks after 12 iterations again in (P) , and then its next blink in (Q) takes 13 iterations. In digraph (A) in Figure 10, this is represented as going through the loop at node (P) twice and then using the edge (PQ) . Note that the diagram (A) lacks loop at node S and edges from S to P or Q , since those sequences cannot be concatenated in such order; the color of v^- at the end and beginning does not match. To explain the use of node weight on S in diagram (A) in Figure 10, consider the string of sequences $(P)(S)(Q)$. After the blink within sequence (P) , v^- blinks for the first time in sequence (S) after 12 iterations, and then again for the second time after 6 iterations within sequence (S) . Then it takes 7 iterations to blink again within sequence (Q) . In terms of diagrams, we walk through the edge weight 12 of (PS) , and then node weight 6 of S , and then edge weight 7 of (SQ) . This shows (i).

For type (b) branches, observe that if v^- blinks as sparse as possible in the dynamics, then it would only use the “long periodic” sequences (I) and (J) , in which case its blinking sequence is generated by diagram (B1) in Figure 10. On the other hand, if v^- blinks as often as

possible, only those four “short periodic” sequences (X)-(W) would be used and its blinking sequence is generated by Figure 10. In general, the actual local dynamics on v and v' could use all combinations, which means that v^- could blink within long periodic sequences (I) and (J) or could skip the second blinks in short periodic sequences (X)-(W). Thus the actual blinking sequence of v^- refines a sequence generated by diagram (B1), but could be coarser than a sequence generated by diagram (B2); skipping second blinks within short periodic sequences corresponds to merging node weights with the following edge weights in diagram (B2). For example, the string (X)(J)(Z) is represented on diagram (B2) by the directed walk $X, (XW), W + (WZ), Z$, which generates the sequence 6, 12, (7+7), 9. This shows (ii) and (iii).

Lastly, suppose v^- only uses sequences (Y) and (Z). Note that the center v does not pull v^- in those sequences, since v^- has colors ≤ 2 whenever the center has color 2. Hence if the induced local orbit on v and v^- is given by an infinite subsequences of (Y) and (Z) only, then the dynamics restricts on $T - T_v$, a contradiction. This shows (iv). \square

Proposition 4.5. *Let (T, X_0) be as before. Let T_w be a fractal branch in T with $w^- \in V(T)$. Then the blinking sequence of w^- refines a sequence generated by digraph (F10-9) or (F11-8) corresponding to the type of T_w .*

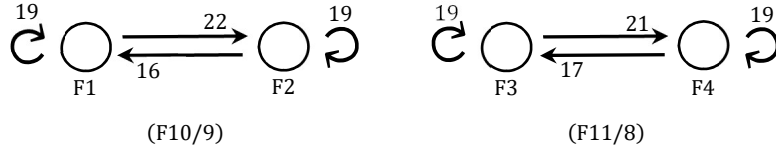


Figure 11: The blinking sequence of w^- refines a sequence generated by (F10/9) or (F11/8) depending on the type of T_w .

Proof. Let T_w be of type 10/9. According to the definition, the dynamic on w and w^- during consecutive blinking gaps of 10 and 9 of w is given by concatenations of the four sequences (F1)-(F4) in Section 3. Note that w^- may or may not blink at some of the a_i 's, b_i 's, c_i 's, or d_i 's. But if one ignores such blinks within each sequence of 19 iterations (Fi)'s, the blinking gap of w^- must be generated by the digraphs in Figure 11. For instance, if sequence (F1) is followed by (F2), then it takes 22 iterations for w^- to blink at the beginning of each (Fi)'s. Thus the actual blinking sequence of w^- must refine a sequence generated by digraphs in Figure 11 depending on the type of T_w . \square

Lemma 4.7. *Let (T, X_0) as before. Suppose $w \in V(T)$ such that each component of $T_w - w$ is either a singleton, branch, or fractal. Then branches of type (a) or (b) or fractal of either types rooted at w are mutually exclusive.*

Proof. Suppose there are both type (a) and (b) branches rooted at w . Then by Proposition 4.4, the blinking sequence of w is generated by the diagram (A) and must refine a sequence generated by diagram (B1) in Figure 10. It is easy to see that the sum of the edge and vertex weights in any directed walk in diagram (A) cannot be 14 or 16. This means that any sequence generated by (A) cannot refine a sequence which contains a term of 14 or 16. But any sequence generated by (a directed closed walk in) diagram (b1) must contain a term of 14 or 16. Hence this is impossible.

Next, suppose there are one branch B and a fractal branch F rooted at w . Suppose B is of type (a). Then the blinking sequence of w must be generated by diagram (A) in Figure 10

and refine a sequence generated by (F10/9) or (F11/8) in Figure 11. First note that there is no way to refine 16 and 21 using the blinking gaps in diagram (A). Hence the blinking sequence of w must refine the constant sequence of gap 19. It remains to show that sequence 19, 19, 19 cannot be refined by any sequence generated by diagram (A). Note that there are 3 ways to refine 19 using weights in diagram (A): $(YX)(XW)$, $(WY)(YY)$, and $(YX)(XX)$ (here we may take $Y = Z$). Notice that none of them uses gap 6 inside sequence (W) or begins with an edge emanating from node X in diagram (A). Hence the first 19 must be refined by $(WY)(YY)$, and the second 19 must be refined by $(YX)(XW)$, but then following 19 cannot be refined. Thus shows that branch of type (a) is exclusive.

Now suppose the branch B is of type (b). A blinking sequence generated by diagram (B2) in Figure 10 must refine a sequence generated by (F10/9) or (F11/8) in Figure 11. To this end, we claim the following: among all directed walks in diagram (B2),

- (a) $Z, (ZX), X$ is the only walk which generates a sequence (9, 7, 6) that refines 22;
- (b) $W, (WY), Y$ is the only walk which generates a sequence (7, 7, 7) that refines 21;
- (c) $(XY), Y$ is the only walk which generates a sequence (10, 7) that refines 17;
- (d) $(XZ), Z$ and $(XW), W$ are the only walks which generate sequences (10, 9 and 12, 7, respectively) refining 19.

To see this, for instance, consider possible ways to refine 22 using diagram (B2). If gap 12 is used, then it must be $22 = 10 + 12$, but 12 cannot be preceded or be followed by 10; if 11 is used, it must be $22 = 11 + 11$, but this is also impossible; if 10 is used, then 12 must be properly refined, but this is impossible; if 9 is used, then $13 = 6 + 7$ is the only way to refine 13, and $Z(ZX)X$ is the only way to generate 6, 7, and 9 consecutively. This shows (a), and the other claims can be shown similarly.

Now we show that we show that any sequence generated by diagram (B2) refines no sequence generated by (F11/8). By (c) and (d), no refinement of 17 can be followed or preceded by any refinement of 19. Since in diagram (F11/8) 19 always follows 17, we see that 17 cannot be refined. This yields that the blinking sequence of w may only refine the constant sequence of 19, but by (d) any refinement of 19 begins with node X and ends with nodes Z or W , so 19 cannot be refined repeatedly.

It remains to show that any sequence generated by diagram (B2) refines no sequence generated by (F10/9). We have seen in the previous paragraph that the constant sequence 19 cannot be refined. So if ever 19 is refined, then at some point a refinement of 22 or 16 should follow. But by (a) and (d), the refinement of 22 cannot follow any refinement of 19. This makes that the directed walk in diagram (F10/9) which generates a sequence refined by some sequence generated from diagram (B2) cannot use the loop at node F1, and consequently, also the right-left edge of weight 16; this implies that only a constant sequence 19 from (F10/9) can be refined, which contradicts our earlier observation. This shows the assertion. \square

Proof of Proposition 4.3. By the minimality, we may assume that the number of distinct phases occupied by the leaves in S is at least 2 and constant in time. At each time t , by a component we mean the set of consecutive states on the leaves on the hexagon \mathbb{Z}_6 ; the size of a component is the number of distinct phases in it. Notice that by Proposition 4.1, whenever ν blinks, every component must lie entirely clockwise or counterclockwise without any leaf opposite to ν . Hence the number of components is a non-increasing functions in time, which must be constant in time since we are in a periodic orbit. Let us call any local configuration in such closed orbit stable.

Figure 9 (a) is the only closed local orbit with a single component of size 3; Figure 9 (d) shows the closed local orbits with a single component of size 2; Figure 9 (c) shows the closed

orbits with 2 components of size 1 and 2; Figure 9 (b) for 2 components of size 1 for both. Notice that any configuration of a component of size ≥ 4 is unstable, likewise any one with two components with one component of size ≥ 3 , any one with two components with both has size ≥ 2 . So the nine configurations in Figure 9 gives all stable local configurations. By the time-invariance types, transitions between local configurations in different types are impossible. Possible transitions within each type and their minimal transition times are investigated similarly in Figure 8. For instance, Figure 12 illustrates possible transitions from Figure 9 b_1 to b_2 . This shows (i).

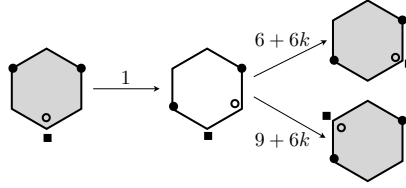


Figure 12: Transitions from Figure 9 b_1 to b_2 . Weights on edges indicate number of iterations, where k is a non-negative integer depending on the dynamics. Due to the rotational symmetry of the targeting configuration, there are two possible phases for v to move in to.

To see (ii), suppose $S = T_v$ is a branch. Note that v^- is the only external neighbor of v , so v can get at most one external pull from v^- in every 6 seconds. This makes the five unshaded local configurations in Figure 9 impossible to appear on branches. For instance, consider possible transitions from the Figure 9 b_4 , which is shown in Figure 13. During the second transition of length 5 from the second to third column, either v^- pulls v as in the dotted bottom transition or not as in the solid upper transition. When v blinks for the next time, none of the resulting configuration in the last column is stable. This shows the bottom left local configuration in Figure 9 is impossible on branches. Similar argument applies for other four unshaded local configurations in Figure 9. Thus there are exactly three possible types of closed orbit for this branch as asserted in (ii).

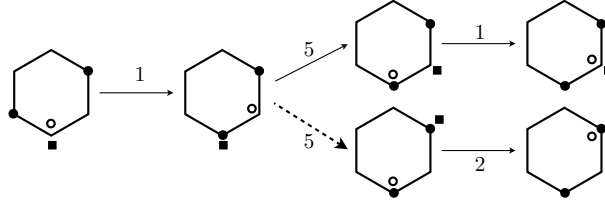


Figure 13: Assuming S is a branch, local configuration in Figure 9 b_4 leads to unstable local configurations.

Now we show (iii). First suppose the local dynamic on T_v is given by Figure 9 (a). In such local orbit, in terms of standard representation, the leaves must have colors 0, 1 and 2 whenever v blinks. The following sequence shows the first 8 iterations starting from such local configuration (Figure 9 a_1):

leaves	012	123	234	345	450	501	012	123
v	2	3	3	3	a_1	a_2	a_3	a_4
v^-	b_1	b_2	b_3	b_4	b_5	b_6		

(5)

Clearly $a_3 \neq 2$, and it is easy to check that $a_3 \neq 5$ leads to a different local configuration at the next blink of v : hence we must have $a_3 = 5$. This requires $2 \in \{b_4, b_5, b_6\}$, which in particular

yields that T_v is open. But $b_4 = 2$ leads to a contradiction since it would yield $b_1 = 5$ and $b_2 = 0$; so $2 \in \{b_5, b_6\}$. We extend sequence (5) as follows:

leaves	012	123	234	345	450	501	012	123	234	345	450	501	012
v	2	3	3	3	a_1	a_2	5	5	5	5	x_1	x_2	x_3
v^-				b_4	b_5	b_6	b_7	b_8	b_9	b_{10}			

Note that $2 \in \{b_5, b_6\}$ yields $b_9 \neq 2$, so $x_1 = 0$, $X_2 = 1$, and $X_3 = 2$. This shows a single transition from Figure 9 a_1 to itself takes exactly 12 seconds, and since $2 \in \{b_5, b_6\}$, T_v is of type (a) definition.

Next, suppose the local dynamic on T_v is given by Figure 9 (b). The argument is similar for type (a). We will show that the transition $b_2 \rightarrow b_1$ and $b_1 \rightarrow b_2$ in Figure 9 takes 9 and 7 seconds, respectively. We look at the first 9 iterations starting from Figure 9 b_2 :

leaves	41	42	53	04	15	20	31	42
v	2	3	3	x_1	x_2	x_3	x_4	x_5
v^-	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8

We need to have $2 \notin \{a_3, a_4, a_5\}$ since otherwise $x_5 = 2$ and the resulting local configuration is not Figure 9 b_1 . This makes $x_3 = 5$ and we may extend the sequence further:

leaves	41	42	53	04	15	20	31	42	53	04	14	25	30	41	52	03	14
v	2	3	3	*	*	5	5	0	1	2	3	b_1	b_2	b_3	b_4	b_5	b_6
v^-	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}	a_{11}	a_{12}	a_{13}	a_{14}	a_{15}	a_{16}	a_{17}

This in particular shows that the transition $b_2 \rightarrow b_1$ in Figure 9 takes 9 seconds. Furthermore, $a_4 \neq 2$ since it leads to a contradiction by back-tracking in time, so we have $2 \in \{a_3, a_5\}$. Hence by definition, T_v would be of type (b) if the transition $b_1 \rightarrow b_1$ in Figure 9 takes 7 seconds, i.e., $b_6 = 2$. To this end, it is enough to show that $2 \notin \{a_{11}, a_{13}, a_{14}\}$. Indeed, $a_{11} \neq 2$ since otherwise $b_6 = 0$ so the local configuration ' $v|leaves$ ' after two more iterations from the end of the above sequence would be 2|30, which is not what we should have as in Figure 9 b_2 . Similarly, $2 \in \{a_{13}, a_{14}\}$ leads to a wrong local configuration 2|30, so $2 \notin \{a_{11}, a_{13}, a_{14}\}$. Thus T_v is of type (b).

Finally, suppose the local dynamic on T_v is given by Figure 9 (c). First five iterations from Figure 9 c_1 is a follows:

leaves	034	134	245	350	401	502
v	2	3	x_1	x_2	x_3	5
v^-	w_1	w_2	w_3	w_4	w_5	w_6

In order for this local dynamics lead to Figure 9 c_2 , we need to have $2 \in \{w_2, w_4, w_5\}$. However, $w_4 = 22$ would lead to a contradiction by back-tracking upto w_1 , so $2 \in \{w_2, w_5\}$. An entirely similar argument for previous cases shows that the transitions $c_1 \rightarrow c_2$ and $c_2 \rightarrow c_1$ in Figure 9 take exactly 10 and 9 seconds, respectively. Thus T_v is fractal of type 10/9. This shows the assertion. ■

5. Proof of Lemma 2.3

By Proposition 9 (iii) we know that type (c) terminal branches are fractal, so in order to show Lemma 2.3, it suffices to show that no terminal branches can be of type (a) or (b). We begin by ruling out type (a) terminal branches.

Proposition 5.1. *Let (T, X_0) be as before. If there are two type (a) terminal branches B and B' rooted at the same vertex w , then one of the two branches must only use the sequence (P), and the other must only use (Q), which are given in the proof of Proposition 4.4.*

Proof. First note that if the blinking sequence of w ever uses the term 11, then because there is only one weight of 11 in Figure 10 (a), both branches undergo the sequence (S) in synchrony. Since w fluctuates the centers of B and B' in the same way, the two branches will be in synchrony thereafter, contradicting the minimality. Thus we may assume that w never have a blinking gap 11. Similarly, we may assume that blinking gap 13 never appears for w . In general, the same argument applies to any unique sequence generated by diagram (A) in Figure 10, such as 7-7, 12-6, and 12-7. Once we exclude such segments, the only possible directed closed walk in Figure 10 (A) is the ones that uses loops on nodes (P), (Q) or (R). Since the induced dynamics on w must coincide, this is possible only if on of the two branches constantly use sequence (P) and the other (R), as asserted. \square

Proposition 5.2. *Let (T, X_0) be as before. Then there is no terminal branch of type (a).*

Proof. Suppose there is a terminal branch T_v of type (a). Then each component in $T_{v^-} - v^-$ is either a leaf or a branch. First suppose there is another branch, say T_u , rooted at v^- . By Lemma 4.7, it must be of type (a) as well. By Proposition 5.1, we may assume that T_v only uses sequence (P) and T_u only (R). Any more branch rooted at w will be redundant. So we assume these two are the only branches rooted at w . Consider the following sequence, which is obtained by overlapping (P) and (R) by matching dynamics on $v^- = u^-$:

$$\begin{array}{cccccccccccccccc} u & 2 & 3 & 3 & 3 & 4 & 5 & 5 & 5 & 5 & 5 & 0 & 1 & 2 & 3 \\ v & 1 & 2 & 3 & 3 & 3 & 4 & 4 & 5 & 5 & 5 & 5 & 0 & 1 & 2 \\ v^- & - & 5 & 5 & 0 & 1 & 2 & 3 & - & - & - & - & - & 5 & 5 \end{array} \quad (\text{PR})$$

Note that since $* \neq 2$, this sequence requires v^- to be pulled four times in 6 iterations when it goes through the $-$'s. Since any vertex blinks at most once in 6 seconds, this means that v^- must have at least 4 external neighbors except v and u . Thus except its own parent v^{--} , it must have at least three leaves. By Proposition 4.3, the local dynamics on this 3-star centered at v^- the local dynamic should be given by Figure 9 (a) or (c). However, the latter is not possible since in our circumstance the blinking sequence of v^- is the constant sequence 12, 12, \dots , which is not the form of $10 + 6k_1, 9 + 6k_2, 10 + 6k_3, \dots$. Thus the 3-star centered at v^- must go through type (a) closed orbit. In particular, whenever v^- blinks, its three leaves must have colors 0, 1, 2. Adding this to $(P + Q)$, we see that the local dynamics on T_{v^-} must be of the concatenation of the following sequence

$$\begin{array}{cccccccccccccccc} u & 3 & 3 & 3 & 4 & 5 & 5 & 5 & 5 & 5 & 0 & 1 & 2 & 3 \\ v & 2 & 3 & 3 & 3 & 4 & 4 & 5 & 5 & 5 & 5 & 0 & 1 & 2 \\ v^- & 5 & 5 & 0 & 1 & 2 & 3 & 3 & - & - & - & 5 & 5 & 5 \\ \text{leaves} & 234 & 345 & 450 & 501 & 012 & 123 & 234 & 345 & 450 & 501 & 012 & 123 & 234 \end{array} \quad (\text{PRI})$$

There are multiple contradictions at this point: v^- still needs to be pulled twice from external neighbors when it goes through $-$'s in the above sequence (PRI) but v^{--} is the only remaining external neighbor; whenever u or v pulls v^- , some leaf pulls v^- together, so the branches T_v and T_u are not contributing anything to the dynamics on v^- , contradicting the minimality.

Hence we may assume that there is no other branch rooted at v^- . Observe that since v^- has color 4 at the end of sequence (S), it must be concatenated with the other three, so sequences (P)-(R) must be used at least once in the periodic local dynamics. Note that v^- must

have color 4 or 5 at the end of sequences (P)-(R) in order to be concatenated by a following one. Since v^- does not blink and v does not pull v^- within those three sequences, it means that v^- must be pulled either by its own parent v^{--} or by its leaves, at least four times during the last six iterations in the three sequences. Since every vertex blinks at most once in every six iterations, this yields that v^- needs to have at least three leaves except its own parent. On this 3-star centered at v^- , Proposition 4.3 again enforces the local dynamics given by Figure 9 (a) (cf. (c) is not the case as before). Thus every blinking gap of v^- should be of the form $12 + 6k$. Since sequence (S) contains a blinking gap 6, it cannot be used and only the other three can be. Now the same sequence (PRI) shows that whenever T_v goes through sequences (P) or (R), v^- is pulled by some leaf whenever it is pulled by v . Since v blinks at exactly same time in sequences (Q) and (R), this means T_v is redundant to the dynamics of v^- , contradicting the minimality. This shows the assertion. \square

Now we rule out type (b) terminal branches.

Proposition 5.3. *Let (T, X_0) as before. Suppose T has a terminal branch T_v of type (b). Then the local dynamic on T_v does not use the long periodic sequences (I) and (J). In particular, the exact blinking sequence of v^- is generated by digraph (B2) in Figure 10.*

Proof. Suppose for contrary that sequences (I) and (J) do appear. This means v^- have blinking gap from one of the four edge weights in Figure 10 (B1). We are going to show that T_v is the only branch rooted at v^- and v^- has at most two leaves. The assertion then easily follows. Indeed, in the last 6 iterations in both sequences (I) and (J), v^- must be pulled at least four times. Since v does not pull v^- during this period, its parent v^{--} and two other leaves cannot provide this.

We first show that v^- has at most two leaves. Suppose not. By Proposition 4.3, the 3-star centered at v^- has local dynamics given by Figure 9 (a) or (c). Suppose the former. Then the blinking sequence of v^- is of the form $12 + 6k_1, 12 + 6k_2, \dots$. Among the weights in Figure 10 (B1), the edge weight 18 of (IJ) is the only one of that form, and the following blinking gap of v^- should be either 16 of (JJ), 14 of (JI), or their refinements. The first two are not of the prescribed form, so it must be their refinements. In Figure 10 (B2), the edge (IJ) in the coarsened diagram (B1) corresponds to the node Y and edge (YW) combined. Thus any refined blinking gap of v^- must use the node weight 7 at W in diagram (B2), which also conflicts with the prescribed form. Hence v^- and its leaves cannot have local dynamics given by Figure 9 (a).

Assuming local dynamics on T_v given by Figure 9 (c), the blinking sequence of v^- must be of the form $10 + 6k_1, 9 + 6k_2, 10 + 6k_3, \dots$. Notice that there is no weight of $9 + 6k$ for $k \geq 1$ in diagram (B1) and (B2) in Figure 10, so the blinking sequence of v^- must be of the form $10 + 6k_1, 9, 10 + 6k_2, \dots$. The only weights of the form $10 + 6k$ in (B1) and (B2) in Figure 10 are 10 and 16. This yields that the blinking sequence v^- must consist of three terms 9, 10, and 16, where 10 and 16 is followed by 9 and 9 must be followed by 10 or 16. We shall see this is impossible. Note that the sequence 10-9 is uniquely generated by the walk (XZ) , Z in Figure 10 (B2), but no edge emanating from node Z in that digraph has weight 10 or 16. Thus 10 is not a blinking gap of v^- , so the blinking sequence must alternate 9 and 16. But such a sequence cannot refine any sequence generated by Figure 10 (B1), a contradiction. This shows that v^- has at most two leaves.

It remains to show that there is no other branch rooted at v^- . Suppose for contrary that another branch T_u is rooted at v^- . By Lemma 4.7, T_u is of type (b). By the minimality, branches T_v and T_u must have distinct dynamics. Now if v^- has blinking gap 14 or 18, then since those gaps are uniquely generated by Figure 10 (B1), the two branches must be synchronized thereafter, a contradiction. Thus v^- never have blinking gaps 14 or 18, but does

use gap 16, which are given by the loops at node I or J in Figure 10 (B1). We may assume that when v^- has blinking gap 16, T_v and T_u undergo loops (II) and (JJ) in Figure 10 (B1), respectively. Note that the loop (II) is represented by the node X and its loop (XX) combined in the refining digraph Figure 10 (B2), so the blinking gap of v^- that follows 16 should be coming from the four edges emanating from node X in the same digraph, which only give 10 or 12. By the parallel reasoning, loop (JJ) must be followed by an edge emanating from nodes Z or W in Figure 10, which yields the next blinking gap should be either 7 or 9, a contradiction. Hence T_v is the unique branch rooted at v^- . This shows the assertion. \square

Proposition 5.4. *Let (T, X_0) as before. Suppose T has a terminal branch T_v of type (b). Then T_v is the only branch rooted at v^- .*

Proof. Suppose for contrary there is another branch T_u rooted at v^- . By Lemma 4.7 we know that T_u must be of type (b), and by Proposition 5.3, they never use long periodic sequences (I) and (J) so that the blinking sequence of v^- is generated by digraph (B2) in Figure 10. By minimality, these two branches must not be synchronized. This means that we must be able to find two distinct closed walks in digraph (B2) which generate the same sequence. Since the weights 6, 11, and 12 are unique in the diagram, any such blinking sequence cannot use those numbers. Thus we may delete the node X together with all the indecent edges, and also the edge (YW) of weight 11 from the digraph. The resulting digraph, which generates the blinking sequence of v^- in our current situation, is provided below:

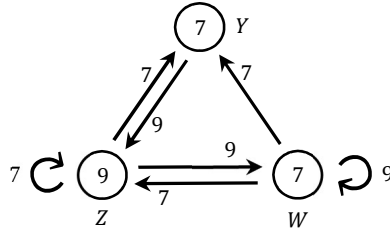


Figure 14: If T_v is a terminal branch of type (b), then the blinking sequence of v^- is generated by this digraph.

Note that by Proposition 4.4 (iv), both branches must use (W) at least once. We claim that the blinking sequence of v^- never repeat 9 twice. This would yield the assertion as follows. Under this assumption, it would be impossible to use the edge (ZW) ; thus no edge heading toward (W) is available, so after a branch uses the node (W) , then it must be confined there. Thus both branches use node W only (recall that we are in a periodic orbit), and since they should generate the same blinking sequence for v^- , they must have synchronized dynamics, a contradiction.

Thus it suffices to show that the blinking sequence of w cannot repeat 9 twice. Suppose for contrary that $g_1(v^-) = g_2(v^-) = 9$. Observe that there are only two ways to generate 9-9 from Figure 10 (B2) with node (X) deleted: $(YZ), Z$ and $Z, (ZW)$. Thus we may assume T_v goes through (YZ) and T_u goes through (ZW) simultaneously. Since the string 7-7-7 is uniquely generated by $W, (WY), Y$ in the above digraph, it never appears in the blinking sequence of v^- . This forces T_u to be confined at node W after the third blink, forcing $(g_i)_{i \geq 3}$ to alternate 7 and 9. This contradicts the periodicity of the blinking sequence, so string 9-9 never appears in the blinking sequence of v^- . This shows the assertion. \square

Proposition 5.5. *Let (T, X_0) as before. Then T has no terminal branch of type (b).*

Proof. Suppose for contrary that T_ν is a terminal branch of type (b). By Proposition 5.4, we know that there is no other branch rooted at ν^- . Thus all neighbors of ν^- except ν and its own parent ν^{--} are leaves. Furthermore, by Proposition 5.3, the local dynamics on T_ν uses only those short periodic sequences (X)-(W) and the blinking sequence of ν^- is generated by digraph (B2) in Figure 10.

We first show that sequence (X) is never used. To see this, notice that in sequence (X), ν^- to be pulled at least four times during the last six iterations. Since ν does not pull ν^- during this period, ν^- must have at least three leaves. By Proposition 4.3, ν^- has exactly three leaves and the 3-star centered at ν^- has local dynamics given by Figure 9 (a) or (c). This implies that the blinking gap of ν^- must always be the form of $12 + 6k_1$, $10 + 6k_2$, $9 + 6k_3$ for $k_1, k_2, k_3 \geq 0$. But sequence (X) forces ν^- to have blinking gap 6, which is not of the prescribed form, a contradiction. Thus sequence (X) is never used by the local dynamics on T_ν . In particular, $g_i(\nu^-) \in \{7, 9, 11\}$ for all $i \geq 1$.

Next, we show that sequences (Y) and (Z) also are not used by the local dynamics on T_ν . Suppose not. Proposition 4.4 (iv), we know that sequence (W) is used (periodically). In particular, ν^- has blinking gap 7 periodically. Observe that sequence (Y) itself requires ν^- to be pulled four times in the last six iterations, and sequence (Z)(W) concatenated also requires the same. Thus ν^- needs to have at least two leaves. Combining Proposition 4.3 together with the conclusion of the previous paragraph and the fact that ν^- does have blinking gap 7, we see that the local dynamics on the star centered at ν^- must only use local configurations Figure 9 b_1 or b_2 , and blinking sequence of ν^- alternates 7 and 9. But then once T_ν uses the node W in digraph (B2) in Figure 10, it must confined on node W, contradicting to the periodicity of local dynamics. Thus T_ν only uses sequence (W), and the blinking sequence of ν^- alternates 7 and 9.

Note that the concatenated sequence (W)(W) requires ν^- to be pulled at least three times at the end of the first (W). Since ν does not pull ν^- during this period, ν^- needs to have at least one leaf. If it has at least two leaves, then combining sequence (W) with the local dynamics on ν^- together with its leaves given by Figure 9 (b), the local dynamics on T_{ν^-} is given by repeating the following sequence

ν	2	3	-	-	-	-	5	0	1	2	3	-	-	5	0	1	2
ν^-	5	5	0	1	2	3	-	-	5	0	1	2	3	-	-	-	-
leaves	20	31	4	53	04	14	25	30	41	52	03	14	24	35	40	51	02

(W1)

Note that in the above sequence, whenever ν blinks, one of the two leaves of ν^- blinks as well. Hence the branch T_ν is redundant to the dynamics of ν^- , which contradicts minimality. So we may assume ν^- has exactly one leaf. The local dynamics on this 1-star centered at ν^- is given by the digraph in Figure 7. The only compatible closed walk there which generates a sequence that alternates 7 and 9 is $e, (ea), a, (ad), d, (dd), d, (de), e$ (upto choice of starting node). Notice that during the loop (ee) , the leaf of ν^- has color 4 when w is blinking. If we plug the leaf in (W), we get the following sequence:

ν	2	3	-	-	-	-	5	0	1	2	3	-	-	5	0	1	2
ν^-	5	5	0	1	2	3	-	-	5	0	1	2	3	-	-	-	-
leaf	0	1	2	3	4	4	5	0	1	2	3	4	4	5	0	1	2

(W1)

which should appear in the local dynamics on T_{ν^-} periodically. Now the last four iteration is conflicting, since the leaf of ν^- does not contribute to extra pull on ν^- . This shows the assertion. \square

6. Proof of Lemma 2.4

In this section we show Lemma 2.4. Let (T, X_0) as before and let T_{v^-} as stated in Lemma 2.4. We say a neighbor of v^- *external* if it is either a leaf or its own parent v^{--} . By Proposition 4.3, v^- has at most three leaves. Hence v^- can have at most four external pulls during every six iterations. Since large blinking gaps of v^- generally require lots of external pulls, it would be not likely under our hypothesis. In fact, blinking gaps of v^- can be at most 11, as stated in the following proposition:

Proposition 6.1. *Let (T, X_0) be as before. Suppose that each connected component of $T_{v^-} - v^-$ is either a singleton or fractal. Further assume that at least one such component which is fractal. Then the blinking gaps of v^- are bounded above by 11.*

Our strategy for showing the above statement is the following: we collect all possible subsequences arising from the two sequences (F1), (F2) and their eight concatenations (Fi)(Fj) for $(i, j) \in \{1, 2\}^2 \cup \{2, 3\}^2$ with respect to the induced blinking gap of v^- , and count the number of required external pulls. A detailed proof of this statement is give at the end of this section.

For further discussions, we give a full list of possible subsequences of (Fi)(Fj) generating a fixed blinking gap of v^- . generating blinking gaps ≤ 11 for v^- below:

(11) Blinking gap 11:

(F1)(a_5)	3	3	–	–	–	–	5	0	1	2	3	–
(F2)(b_8)	–	–	–	5	0	1	2	3	–	–	–	–
(F3)(c_6)	–	–	–	–	–	–	5	0	1	2	3	–
(F4)(d_8)	–	–	–	–	5	0	1	2	3	–	–	–
(a_2)(F1)	1	2	3	–	–	–	–	5	0	1	2	3
(a_5 or b_5)(F2)	–	–	–	–	5	0	1	2	3	–	–	–
(c_3 or d_3)(F3)	1	2	3	–	–	–	5	0	1	2	3	–
(c_5 or d_5)(F4)	3	–	–	–	5	0	1	2	3	–	–	–
v^-	2	3	x_1	x_2	x_3	x_4	x_5	x_6	5	0	1	2

(10) Blinking gap 10:

(F1)(a_4)	3	3	–	–	–	–	5	0	1	2	3
(F2)(b_7)	–	–	–	5	0	1	2	3	–	–	–
(F3)(c_5)	–	–	–	–	–	–	5	0	1	2	3
(a_3 or b_3)(F1)	2	3	–	–	–	–	5	0	1	2	3
(c_4 or d_4)(F3)	2	3	–	–	–	5	0	1	2	3	*
(c_6 or d_6)(F4)	–	–	–	5	0	1	2	3	–	–	–
v^-	2	3	x_1	x_2	x_3	x_4	x_5	5	0	1	2

(9) Blinking gap 9:

(F1)(a_3)	3	3	–	–	–	–	5	0	1	2
(F3)(c_4)	–	–	–	–	–	–	5	0	1	2
(F4)(d_6)	–	–	–	–	5	0	1	2	3	–
(a_4 or b_4)(F1)	3	–	–	–	–	5	0	1	2	3
(a_7 or b_7)(F2)	–	–	5	0	1	2	3	–	–	–
(c_5 or d_5)(F3)	3	–	–	–	5	0	1	2	3	–
v^-	2	3	x_1	x_2	x_3	x_4	5	0	1	2

(8) Blinking gap 8:

(F1)(a_2)	3	3	–	–	–	–	5	0	1
(F2)(b_5)	–	–	–	5	0	1	2	3	–
(F3)(c_3)	–	–	–	–	–	–	5	0	1
(F4)(d_5)	–	–	–	–	5	0	1	2	3
(a_5 or b_5)(F1)	–	–	–	–	5	0	1	2	3
(a_8 or b_8)(F2)	–	5	0	1	2	3	–	–	–
(c_6 or d_6)(F3)	–	–	–	5	0	1	2	3	–
(c_8 or d_8)(F4)	–	5	0	1	2	3	–	–	–
ν^-	2	3	x_1	x_2	x_3	5	0	1	2

(7 and 6) Blinking gaps 7 and 6:

(F1)(a_1)	3	3	–	–	–	–	5	0		(F2)(b_3)	–	–	–	5	0	1	2
(F2)(b_4)	–	–	–	5	0	1	2	3		(F4)(d_3)	–	–	–	–	5	0	1
(F3)(c_2)	–	–	–	–	–	–	5	0		(a_1)(a_7)	0	1	2	3	–	–	–
(F4)(d_4)	–	–	–	–	5	0	1	2		(a_2)(a_8)	1	2	3	–	–	–	–
(a_1)(a_8)	0	1	2	3	–	–	–	–		(c_2)(c_8)	0	1	2	3	–	–	–
ν^-	2	3	x_1	x_2	5	0	1	2		ν^-	2	3	4	5	0	1	2

For instance, (F1)(a_5) is the subsequence of (F1) from the first blink of ν^- to the second blink $a_5 = 2$; (a_2)(F1) is the subsequence of (F1)(F1) from $a_2 = 2$ to the first blink of ν^- in the second (F1). Note that the last three sequences for gap 6 are contradictory, so only the first two are possible.

Proposition 6.2. *Let (T, X_0) be as before. Suppose that each connected component of $T_{\nu^-} - \nu^-$ is either a singleton or fractal. Further assume that at least one such component which is fractal. Then ν^- has at least two leaves.*

Proof. Suppose for contrary that ν^- has at most one leaf. By Proposition 6.1, blinking gaps of ν^- are at most ≤ 11 . Note that blinking gap 10 is impossible, since it requires at least three external pulls during the first six iterations while ν^- has at most two external neighbors.

We first claim that ν^- needs to have at least two leaves in order to have blinking gap 9. Since gap 9 requires at least one leaf for ν^- , we may assume for contrary that ν^- has exactly one leaf. Note that (a_7 or b_7)(F2) is necessary to generate gap 9 with ν^- since otherwise blinking gap 9 would require at least three external neighbors. Among the sequences which generate gaps $\in \{6, 7, 8, 11\}$, (F2)(b_5) for gap 8 is the only sequence which can follow. Similarly, the sequence (b_5)(F1) for gap 8 can only follow this. Hence we only need to rule out consecutive gaps 8-8. Note that in Figure 7, $a \rightarrow c \rightarrow d$ and $e \rightarrow b \rightarrow d$ are the only possible walks that generate blinking sequence 8-8 for ν^- . But note that during the second transition in each walk, the center is not pulled by the leaf. This makes the second blinking gap 8 for ν^- during (b_5)(F1) impossible. This shows the second claim. Thus we may assume that $g_i(\nu^-) \in \{6, 7, 8, 11\}$ for all $i \geq 1$.

Our second claim is that the assertion holds assuming ν^- has blinking gaps ≤ 8 only. By ruling out sequences above which cannot be concatenated by any other to the right or left, we find that the local dynamics of ν should be given by repeating the following sequences:

$$(a_8)(F2) - (F2)(b_5) - (b_5)(F1) - (F1)(a_1) - (a_1)(a_8); \quad 8 - 8 - 8 - 7 - 7$$

which induce stings of blinking gaps of ν^- as indicated on the right. By the asymmetry of such strings and minimality, this yields that T_ν is the unique fractal subtree rooted at ν^- . However,

this means that whenever v^- has blinking gap 8 induced by sequence (F2)(b_5), in which v does not pull v^- , v^- needs two external pulls, a contradiction. This shows the second claim.

Now we show the assertion. If v^- has no leaf, then blinking gap 11 is impossible so the assertion follows from the two claims. Hence we may assume that v^- has one leaf. By Proposition 4.3, the blinking sequence of v^- is generated by the digraph in Figure 9. In fact, only the edges or loops with weights in $\{6, 7, 8, 11\}$ can be used. Moreover, edges of weight 11 or 8 emanating nodes b or c in the digraph cannot be used, since in the first six iterations starting from those local configurations v^- is not pulled by the leaf. Deleting all those edges, we obtain the following digraph which should generate the blinking sequence of v^- :

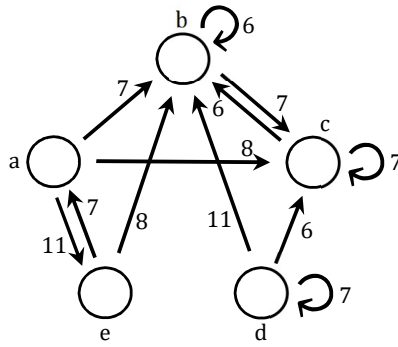


Figure 15: A digraph generating the blinking sequence of v^- when it has a single leaf under the hypothesis of Proposition 6.2.

By the second claim, v^- must have blinking gap 11, and the only closed walk in the above digraph containing an edge of weight 11 is the one alternating between node a and e . Hence the blinking sequence of v^- should alternate 11 and 7, but no sequences for gap 7 can be followed by any sequence for 11. This shows the assertion. \square

Proposition 6.3. *Let (T, X_0) be as before. Suppose that each connected component of $T_{v^-} - v^-$ is either a singleton or fractal. Further assume that at least one such component which is fractal. Then the blinking sequence of v^- alternates 10 and 9 or 11 and 8.*

Proof. By Proposition 6.1, we know that the blinking gaps of v^- are at most 11. We first show that the blinking sequence of v^- alternates 8 and 11 or 9 and 10. By Proposition 6.2, v^- has at least two leaves. By Proposition 4.3, v^- never have blinking gap 6. If v^- has three leaves, then by Proposition 4.3 the local dynamics on the 3-star centered at v^- should be given by Figure 9 (c), so v^- has blinking gap alternating 10 and 9. Hence we may assume that v^- has exactly two leaves. Next, we show that string 9-9 also cannot appear in the blinking sequence of v^- . The two leaves of v^- forces that the following blinking gap of v^- after 9-9 should be either 10 or 11. From the list of subsequences generating blinking gap 9, we see that v^- must blink at a_3 or c_4 at the end of second blinking gap 9. But no subsequence for gap 11 begins with these, and the fourth and fifth subsequence for gap 10 do begin with these but during which v^- is not pulled by v : this requires v^- to have four external pulls during a blinking gap 10, a contradiction.

Next, we rule out blinking gap 7 for v^- . Suppose v^- does have blinking gap 7. Then the local dynamics on the 2-star centered at v^- is given by digraph (b) in Figure 9. Moreover, since gap ≥ 12 and 9-9 does not appear, the only possible local configurations for this 2-star are Figure 9 b_3 and b_4 . This forces that gap 7 is always followed and preceded by gap 9. Observe

that for sting 9-7-9 in the blinking sequence of v^- , by considering possible concatenations of the subsequences in Figure 2.3, one see that the second 9 after 7 should be given by $(b_4)(F1)$. This uniqueness forces that T_{v^-} is the only fractal subtree rooted at v^- . Moreover, the second 9 cannot be followed by 7, since otherwise the second 7 is given by $(F1)(a_2)$, but no sequence for 9 begins with (a_2) . Thus the blinking sequence of v^- must contain the string 9-7-9-10-9-7-9. However, the second 7 in this string must end with (a_1) , but no sequence for gap 9 begins with (a_1) . Thus v^- does not have blinking gap 7 if it has at least two leaves.

Now we may assume that v^- has exactly two leaves only the gaps 8,9,10, and 11 appears. If the 2-star centered at v^- has local dynamics confined in digraphs (b) or (c) in Figure 9, then we are done. Hence we may assume that the local dynamics are given by digraph (d) in Figure 9. We want to show that the local configuration on the 2-star alternates local configurations d_2 and d_3 in Figure 9 (d). Since we have shown that 9-9 does not appear in the blinking sequence of v^- , it suffices to rule out the string 8-9-11 and 11-11. First, observe that the former is uniquely generated by $(F4)(d_5)-(d_5)(F3)-(F3)(c_6)$. Hence v^- has at most one fractal branch rooted at it, and needs at least four external pulls during the last blinking gap of 11, a contradiction. To rule out the string 11-11, observe that there are five sequences that generates consecutive blinking gap 11 of v^- :

$$\begin{aligned} & (F1)(a_5)-(a_5)(F2) \\ & (a_2)(F1)-(F1)(a_5) \\ & (a_5 \text{ or } b_5)(F2)-(F2)(b_8) \\ & (c_3 \text{ or } d_3)(F3)-(F3)(c_6) \\ & (c_5 \text{ or } d_5)(F4)-(F4)(d_4) \end{aligned}$$

Since the blinking sequence of v^- is generated by the digraph (d) in Figure 4.3, the next blinking gap after 11-11 should be either 9 or 11. Note that no subsequences generating those blinking gaps could be concatenated after the last three sequences above. This yields that v^- could have at most two fractal subtrees rooted at itself whose local dynamics during the second blinking gap 11 should be given by subsequences $(a_5)(F2)$ or $(F1)(a_5)$. But then the second blinking gap 11 of v^- requires at least four external pulls, which is impossible with only two leaves for v^- . This shows the assertion. \square

Now we are ready to prove Lemma 2.4.

Proof of Lemma 2.4. By Propositions 6.1, 6.2, and 6.3, we may assume that v^- has at least two leaves and its blinking sequence alternates 10 and 9 or 11 and 8. To show that T_{v^-} is fractal, we need to show that $v^{--} \in V(T)$ and it provides external pulls on v^- at right place. First let us analyze the blinking sequence that alternates 8 and 11. In this case, by Proposition 4.3 we may assume that v^- has exactly two leaves. By Proposition 4.3, the 2-star centered at v^- should alternate between the two local configurations in Figure 9 d_2 and d_3 . Hence when v^- blinks at the beginning of a gap 11, its two leaves must have colors 3 and 4. Consider the following 11 iterations during a gap 11:

v^-	2	3	x_1	x_2	x_3	x_4	x_5	x_6	5	0	1	2
leaves	34	34	45	50	01	12	23	45	50	01	12	23
v^{--}	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8				

From the list of possible sequence for gap 11, v^- is not pulled by any of its internal neighbors (i.e., centers of fractal subtrees rooted at v^-) during the transition $x_1 \rightarrow x_5$. In order to make gap 11, v^- needs to be pulled by an external neighbor during $x^1 \rightarrow x^3$. Thus $v^{--} \in V(T)$ and $2 \in \{y_2, y_3, y_4\}$. Since $y_4 = 2$ yields $y_1 = 5$ which is a contradiction, we have $2 \in \{y_2, y_3\}$. This shows T_{v^-} is fractal of type 11/8, as desired.

Now we assume that the blinking sequence of v^- alternates 10 and 9. The argument is similar as before. By Proposition 4.3, the k -star ($k \in \{2, 3\}$) centered at v^- should alternate local configurations b_1 and b_2 or c_1 and c_2 in Figure 9. In the first case, v^- has two leaves which have colors 0 and 4 when v^- blinks at the beginning of gap 10 as in Figure 4.3 b_2 ; in second case it has three leaves of color 0, 3, and 4 at such instant as in Figure 9 c_1 . Following sequence shows ten iterations during a blinking gap 10 together with all possible local dynamics on the leaves of v^- :

v^-	2	3	x_1	x_2	x_3	x_4	x_5	5	0	1	2
leaves	04	14	25	30	41	02	13	25	30	41	52
leaves	034	134	245	350	401	012	123	235	340	451	502
v^{--}	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8			

By the list of sequences giving blinking gap 10 for v^- , there is no internal pulls on v^- during the first six iterations in the above sequence. Hence v^- still needs one extra external pulls, and this yields $v^{--} \in V(T)$ with $2 \in \{y_2, y_4, y_5\}$. Since $y_4 \neq 2$ for similar reason this shows that T_{v^-} is fractal of type 10/9. This shows the assertion. ■

Proof of Proposition 6.1. By Proposition 4.5, the maximum possible blinking gap of v^- is 22 generated by (F1)(F2) without secondary blink within (F1), but this requires at least 5 external pulls during six iterations, so it cannot occur. Blinking gap 20 arises from (F3)(F4) but impossible for similar reason, and there is no subsequence which gives blinking gap 20 (e.g., see Figure 11). We rule out large blinking gaps from 19 to 12 below.

- (19) There are only four subsequences giving gap 19, namely, $(Fi)(Fi)$ for $1 \leq i \leq 4$ without v^- blinking more than once in the first sequence (Fi) . Hence if v^- has blinking gap 19, then there can be at most four fractal subtrees rooted at v^- . We overlap all four sequences to see the least number of required external pulls:

(F1)(F1)	3	3	–	–	–	–	5	0	1	2	3	–	–	–	–	5	0	1	2	3
(F2)(F2)	–	–	–	5	0	1	2	3	–	–	–	–	5	0	1	2	3	–	–	–
(F3)(F3)	–	–	–	–	–	–	5	0	1	2	3	–	–	–	5	0	1	2	3	–
(F4)(F4)	–	–	–	–	5	0	1	2	3	–	–	–	5	0	1	2	3	–	–	–
v^-	2	3	x_1	x_2	x_3	x_4	x_5										5	0	1	2

where row $(Fi)(Fi)$ above indicates the dynamics of the center of a fractal subtree of sequence $(Fi)(Fi)$. Since v^- does not blink during gap 19 and we have all possible fluctuations from the fractal subtrees in the above matrix, all missing pulls on v^- must be external. Now note that $3 \leq x_5 \leq 5$, so v^- must be get at least three external pulls during the first six iterations above. This requires v^- to have at least two leaves, but blinking gap 19 is impossible to be generated from digraphs in Figure 9, contradicting Proposition 4.3. This rules out blinking gap 19.

- (18) Impossible

- (17) Gap 17 arises uniquely from (F4)(F3), so in this case v^- can have at most one fractal subtree. In the following sequence during blinking gap 17, v^- needs to get at least 5 external pulls during the first six iterations, which is impossible.

(F4)(F3)	–	–	–	–	5	0	1	2	3	–	–	–	5	0	1	2	3	–
ν^-	2	3	x_1	x_2	x_3	x_4	x_5	x_6							5	0	1	2

- (16) Gap 16 arises uniquely from (F2)(F1), so ν^- can have at most one fractal subtree. Consider the following sequence during blinking gap 16:

(F2)(F1)	*	*	*	5	0	1	2	3	*	*	*	*	5	0	1	2	3
ν^-	2	3	x_1	x_2	x_3	x_4	x_5							5	0	1	2

If $x_5 = 5$, then ν^- needs five external pulls in a row after x_5 , which is impossible. Thus $3 \leq x_5 \leq 4$. Hence ν^- needs at least four external pulls for the first six iterations in the above sequence. So it has exactly three leaves. By Proposition 4.3, the 3-star centered at ν^- must have local configuration Figure 9 c_1 in order to match with blinking gap 16. Inserting the three leaves according to such local configuration, the first six iterations in the above sequence looks as follows:

(F2)(F1)	—	—	—	5	0	1	2
ν^-	2	3	3	3	x_3	x_4	x_5
leaves	012	123	234	345	450	501	012

But since $x_5 \leq 4$, this requires ν^- to be pulled by its only remaining external neighbor, namely its parent, twice for the last three iterations, a contradiction.

- (15) Impossible.

- (14) Within the sequence (F1), we could have blinking gap 14 if $a_7 = 2$. Also possible is $a_2 = 2$ and (F1) is followed by (F2). Last possibility is that $c_1 = 2$ in (F3) and (F4) follows. Denote these three cases by (F1)(a_7), (a_2)(F2), and (c_3)(F4). This gives following sequence for blinking gap 14:

(F1)(a_7)	3	3	—	—	—	—	5	0	1	2	3	—	—	—	—
(a_2)(F2)	1	2	3	—	—	—	—	5	0	1	2	3	—	—	—
(c_3)(F4)	0	1	2	—	—	—	—	5	0	1	2	3	—	—	—
ν^-	2	3	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	5	0	1	2

Hence ν^- requires at least four external pulls during the five iterations from x_2 to x_7 . Hence ν^- has exactly three leaves. But neither digraphs (a) or (c) in Figure 9 can generate gap 14, a contradiction.

- (13) There are four possibilities for gap 13 as below, which clearly requires ν^- to have at least two leaves. But blinking gap 13 is not generated from any digraphs in Figure 9, contrary to Proposition 4.3.

(F1)(a_6)	3	3	—	—	—	—	5	0	1	2	3	—	—	—
(F3)(c_8)	—	—	—	—	—	—	5	0	1	2	3	—	—	—
(c_3)(F4)	1	2	3	—	—	—	5	0	1	2	3	—	—	—
(d_3)(F4)	1	2	3	—	—	—	5	0	1	2	3	—	—	—
ν^-	2	3	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	5	0	1	2

- (12) ν^- could have at most six fractal subtrees generating blinking gap 12. First observe that ν^- cannot have three leaves. To see this, note that by Proposition 4.3 its blinking gaps should be of the form $12 + 6k_i$ for $k_i \geq 0$; since we have seen that ν^- does not have a blinking gap 18, its blinking gap should be 12 constantly. But a fractal subtree rooted at ν^- makes this impossible (e.g., no subsequence is possible between the six possibilities below). Second, suppose ν^- has two leaves. Then by Proposition 4.3, the 2-star centered

at ν^- must have local configuration Figure 9 b_1 or b_3 . Their dynamics are inserted in the following matrix:

$(a_1)(F1)$	0	1	2	3	–	–	–	–	5	0	1	2	3
$(a_4)(F2)$	3	–	–	–	–	5	0	1	2	3	–	–	–
$(b_4)(F2)$	3	–	–	–	–	5	0	1	2	3	–	–	–
$(c_2)(F3)$	0	1	2	–	–	–	–	5	0	1	2	3	–
$(c_4)(F4)$	2	3	–	–	–	5	0	1	2	3	–	–	–
$(d_4)(F4)$	2	3	–	–	–	5	0	1	2	3	–	–	–
ν^-	2	3	x_1	x_2	x_3	x_4	x_5	x_6	x_7	5	0	1	2
leaves (b_1)	04	14	25	30	41	52	03	14	25				
leaves (b_3)	02	13	24	35	40	51	02	13	24				

Note that we cannot have both of the last two rows at the same time. Considering each cases separately, we see that ν^- still needs at least two external pulls, a contradiction. The above matrix also shows that ν^- needs at least two leaves, so blinking gap 12 is impossible.

This shows the assertion. ■

7. Concluding remarks

For each $\kappa \geq 3$, define $\phi(\kappa)$ to be the smallest possible maximum degree of a minimal counterexample (T, X_0) for given κ . In the introduction we have noted that $\phi(\kappa) \leq \kappa$, and in Theorem 3 we have shown that $\phi(\kappa) = \kappa$ for $\kappa \in \{3, 4, 5, 6\}$ but the counterexamples in Appendix A indicates that $\phi(\kappa) < \kappa$ for $\kappa \geq 7$. Our theorem on finite paths (Theorem 2 in [13]) gives $\phi(\kappa) \geq 3$. It would be interesting to see the asymptotic behavior of this cutoff function. For instance, the table in Appendix A seems to indicate that there is a positive constant $\delta > 0$ such that $\phi(\kappa) \geq \delta\kappa$ for large κ , but a priori it is not even clear that ϕ should be unbounded. This question is left for a future work.

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A. Appendix: Non-synchronizing examples for Theorem 3 (ii)

κ	Center(s)	Leaves	period
7	2	0 1 4 5	22
8	5	1 5 7	60
	6	0 3 6	
9	8	0 1 4 5 6	28
10	6	0 1 4 5 7 8	51
11	7	1 4 5 6 7 9 10	34
	2	0 0 1 1 1 1 4 6 6 7	57
12	11	2 3 6 7 9 10 11	61
	6	1 4 5 8 9	62
13	3	0 1 3 7 9 11 12	40
14	10	0 2 5 9 11 13	71
	3	2 3 7 8 11 12	57
	11	0 2 4 9 10	72
	11	2 4 7 12 13	101
15	11	0 1 2 3 9 10 13 14	46
	9	0 4 7 10 11 14	77
16	13	0 1 2 3 6 7 12 13 14	81
	8	3 4 8 9 11 14	82
17	6	0 1 2 5 9 10 12 13 16	52
18	11	0 5 6 11 12 17	73
	17	1 2 7 8 9 11 12 13 14 15	91
	11	3 4 5 5 8 11 16 17	92
	9	1 2 2 4 5 7 8 11 12 14 15	146
	16	0 3 6 10 13	185
19	10	2 3 4 5 6 8 9 12 13 16	58
20	18	2 3 4 6 7 8 9 10 11 15 16	101
	0	2 3 6 7 8 9 15 17	102
21	12	0 3 4 5 8 9 10 13 15 16 17	64
	3	0 1 6 9 10 13 17 20	129
	9	0 1 5 7 11 14 15 19	107
22	1	0 6 7 8 13 14 20 21	89
	11	2 3 9 10 12 14 15 16 17 18 19	111
	9	5 6 7 13 14 18 19 21	112
	9	0 4 6 7 8 9 12 13 14 16 17 21	178
23	17	2 4 9 10 11 14 15 16 17 18 19 20	70
	18	0 3 5 10 15 18	259
	6	1 2 8 9 10 14 15 16 17	117
24	20	1 2 8 9 15 16 17 18 23	122
	23	5 7 12 13 14 15 21 23	97
	21	0 3 4 5 8 10 12 13 14 17 18 19 22	267
25	14	0 4 5 7 7 10 11 12 15 16 19 20 21	76
	20	3 4 5 10 14 15 22 23 24	127
26	12	2 3 4 5 10 13 14 15 21 24	132
27	25	3 4 9 10 11 12 13 21 22 25	137
	12	0 1 7 8 13 14 18 23 24 25	165
28	20	0 1 2 3 11 12 16 21 22 27	242
29	24	2 3 5 6 10 14 15 17 18 21 24 26 28	88
30	18	3 4 8 9 10 15 16 17 21 26 28	152

Figure A.1: A non-exhaustive list of non-synchronizing κ -colorings on trees with maximum degree $< \kappa$ for $7 \leq \kappa \leq 30$. The underlying tree is a star with $\leq \kappa - 1$ leaves for $\kappa \neq 8$; for $\kappa = 8$ the it is obtained by joining centers of two stars with 3 leaves by an edge. Every 8-coloring synchronizes on a star with ≤ 7 leaves.

References

- [1] Arora, A., Dolev, S., Gouda, M., 1992. Maintaining digital clocks in step. In: Distributed Algorithms. Springer, pp. 71–79.
- [2] Boulinier, C., Petit, F., Villain, V., 2006. Toward a time-optimal odd phase clock unison in trees. In: Stabilization, Safety, and Security of Distributed Systems. Springer, pp. 137–151.
- [3] Bramson, M., Griffeath, D., 1989. Flux and fixation in cyclic particle systems. *The Annals of Probability*, 26–45.
- [4] Buck, J. B., 1938. Synchronous rhythmic flashing of fireflies. *The Quarterly Review of Biology* 13 (3), 301–314.
- [5] Dolev, S., 2000. Self-stabilization. MIT press.
- [6] Dorfler, F., Bullo, F., 2012. Synchronization and transient stability in power networks and nonuniform kuramoto oscillators. *SIAM Journal on Control and Optimization* 50 (3), 1616–1642.
- [7] Enright, J. T., 1980. Temporal precision in circadian systems: a reliable neuronal clock from unreliable components? *Science* 209 (4464), 1542–1545.
- [8] Gravner, J., Lyu, H., Sivakoff, D., 2016. Limiting behavior of 3-color excitable media on arbitrary graphs. In preparation.
- [9] Greenberg, J. M., Hastings, S., 1978. Spatial patterns for discrete models of diffusion in excitable media. *SIAM Journal on Applied Mathematics* 34 (3), 515–523.
- [10] Herman, T., Ghosh, S., 1995. Stabilizing phase-clocks. *Information Processing Letters* 54 (5), 259–265.
- [11] Hong, Y.-W., Scaglione, A., 2005. A scalable synchronization protocol for large scale sensor networks and its applications. *Selected Areas in Communications, IEEE Journal on* 23 (5), 1085–1099.
- [12] Lamport, L., 1978. Time, clocks, and the ordering of events in a distributed system. *Communications of the ACM* 21 (7), 558–565.
- [13] Lyu, H., 2015. Synchronization of finite-state pulse-coupled oscillators. *Physica D: Nonlinear Phenomena* 303, 28–38.
- [14] Lyu, H., 2016. Phase synchronization of pulse-coupled excitable clocks. arXiv preprint arXiv:1604.08381.
- [15] Lyu, H., Sivakoff, D., 2016. Synchronization of finite-state pulse-coupled oscillators on \mathbb{Z} . In preparation.
- [16] Mesbahi, M., Egerstedt, M., 2010. Graph theoretic methods in multiagent networks. Princeton University Press.
- [17] Nair, S., Leonard, N. E., 2007. Stable synchronization of rigid body networks. *Networks and Heterogeneous Media* 2 (4), 597.
- [18] Pagliari, R., Scaglione, A., 2011. Scalable network synchronization with pulse-coupled oscillators. *Mobile Computing, IEEE Transactions on* 10 (3), 392–405.
- [19] Strogatz, S. H., 2000. From kuramoto to crawford: exploring the onset of synchronization in populations of coupled oscillators. *Physica D: Nonlinear Phenomena* 143 (1), 1–20.
- [20] Strogatz, S. H., 2001. Exploring complex networks. *Nature* 410 (6825), 268–276.
- [21] Wang, Y., Nunez, E., Doyle, F. J., 2013. Increasing sync rate of pulse-coupled oscillators via phase response function design: theory and application to wireless networks. *Control Systems Technology, IEEE Transactions on* 21 (4), 1455–1462.
- [22] Wang, Y., Núñez, E., Doyle III, F. J., 2012. Energy-efficient pulse-coupled synchronization strategy design for wireless sensor networks through reduced idle listening. *Signal Processing, IEEE Transactions on* 60 (10), 5293–5306.